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ALTERNATING-CURRENT CIRCUITS

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ALTERNATING-CURRENT CIRCUITS

Alternating-Current Circuits

EARLE M. MORECOCK
Supervisor of Electrical Department
Rochester Institute of Technology

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ALTERNATING-CURRENT CIRCUITS

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PREFACE

This textbook is one of a series prepared by the faculty of Rochester Athenaeum and Mechanics Institute as part of its program for developing teaching materials which are functional and closely related to the requirements of industry. With the guidance of W. W. Charters, a continuous study is made of the factors essential to success in various fields. This process of analysis enables the selection of subject matter having a high degree of validity in terms of the definite occupational objectives for which training is offered.

The principal purpose in preparing this book has been to provide a text in alternating-current circuits for use in technical institutes, junior colleges, and industrial and extension schools. It should also be of value to those who have to depend on self-study to further their education. A companion volume by the author, *Direct Current and Magnetic Circuits*, is included in this series.

In schools for which this text has been prepared, the course in alternating-current circuits usually is scheduled before the student has obtained a mathematical background which includes calculus. For this reason, calculus has been avoided in this presentation of the subject. However, many of the concepts, equations, and constants pertaining to alternating-current circuits, which customarily are explained with calculus, are not omitted, but rather are explained graphically. It has been assumed that the student will possess a working knowledge of algebra and trigonometry, and that he will have an understanding of direct-current and magnetic circuit principles before attempting to use this text.

Examples with their detailed solutions, followed by problems of a similar nature, are inserted at frequent intervals throughout the text. These enable the student to evaluate his ability to read and understand the material which has just preceded the problem. Other problems are placed at the ends of the chapters for review. Adequate duplication of problems of similar types is provided so that optional problem assignments can be made.

The laboratory instruction sheets, inserted at the end of each chapter, outline experimental work which is closely related to the subject matter contained in the chapter. The equipment and instruments required for the experimental work are to be found in most school laboratories. It is an established practice at the Rochester Athenaeum and Mechanics Institute to give students a mastery test on each chapter before starting the work of a subsequent chapter. This practice is recommended.

The author wishes to acknowledge and thank Messrs. Earl C. Karker and Clarence E. Tuites, both instructors in the Electrical Department at Rochester Athenaeum and Mechanics Institute, for their assistance in the preparation of this volume.

EARLE M. MORECOCK

Rochester Athenaeum and Mechanics Institute
Rochester, New York
October, 1942

SUGGESTIONS FOR STUDY AND LABORATORY PROCEDURE

Circuit Diagrams. Electric circuit problems which frequently appear to be complicated when expressed in words often become relatively simple when viewed as circuit diagrams. It is recommended, therefore, that whenever possible, a circuit diagram be drawn as the first step in solving a problem. Usually a freehand sketch is adequate. Symbols for voltages, currents, resistances, reactances, and so forth, should be labeled on the diagram. In many cases it may be advisable to transfer the information given in the problem to the diagram.

Vector Diagrams. In dealing with many alternating-current circuit problems, the construction of a vector diagram to supplement the circuit diagram is an additional aid in obtaining a clear understanding of the problem. Generally speaking, the technique of first constructing a circuit diagram and a vector diagram for an a-c problem is equivalent to reducing the problem to little more than a mathematical computation. Vector diagrams drawn freehand with the vectors properly labeled usually are adequate unless graphical methods are to be used for the complete solution of the problem. In the case of polyphase problems it is recommended that the vectors be identified by double-subscript notations.

Preparation for Laboratory Work. The student should study thoroughly the laboratory instruction sheets for each experiment before proceeding with work in the laboratory. He should have clearly in mind the purpose of the experiment, the equipment to be used, and the procedure to be followed. It is suggested that a circuit diagram and a data sheet with columns properly labeled be submitted for the instructor's approval before the experiment is started. Some instructors may request a preliminary report which should include these items. A supply of graph paper, data sheets, and a slide rule should be part of the student's laboratory equipment, and to the extent possible, the experimental data should be checked before the test set-up is dismantled.

Recommended Laboratory Techniques.

1. Plan and arrange the test apparatus, controls, and instruments so that a minimum of motion and delay is required to obtain the data.

2. Check the current and voltage limitations of equipment and instruments, and see that the protective devices have the proper ratings or settings.

3. Before recording data, run through the experiment rapidly to determine whether or not current and voltage values are within safe limits, and whether or not these values can be measured accurately with the instruments that have been selected. Alternating-current ammeters and voltmeters of the types commonly used are seldom accurate when indicating less than 20 per cent of their full-scale range. Should it become advisable to change instruments during a test, the substitute instrument should be readily available so that the change can be made with minimum delay.

4. The pointer of a voltmeter or an ammeter moves beyond the full value of the scale when the magnitude of the quantity to be measured exceeds the rating of the instrument. This is not necessarily the case with a wattmeter, and it is possible for either the potential or the current element of a wattmeter or both to be overloaded and damaged without the pointer indicating excess load. Because of this, the student should acquaint himself with the maximum potential and current limits of each wattmeter before use, and should take care to see that these limits are not exceeded during the test. The maximum ratings of wattmeters are usually marked on the scale of the instrument or are given on a card placed in the instrument case.

5. The data should be checked by approximate calculations or by plotting graphs to make reasonably certain that the results are satisfactory before the circuit is dismantled.

ALTERNATING-CURRENT CIRCUITS

CHAPTER I

ALTERNATING-CURRENT WAVES AND QUANTITIES

1. Use of Alternating Current. In order to transmit large amounts of electrical power efficiently over long distances, high-voltage transmission is essential. The voltage of direct-current systems is limited by commutation to approximately 1500 volts per generator, except in small specially designed machines. The highest voltage d-c transmission of any considerable size in this country is the 3000-volt system of the Chicago, Milwaukee, and St. Paul Railroad. This voltage is produced by connecting two d-c generators in series. It is common practice to build alternating-current generators (alternators) which generate voltages as high as 16,500 volts. The alternator voltage can be stepped up by transformers for transmission and then reduced by step-down transformers at the load to any voltage suitable for industrial or commercial purposes. Transformers have no moving parts, operate at efficiencies as high as 99 per cent, and have a low cost when compared to rotating machinery of equal capacity. Alternating-current transmission systems at potentials of 212,000 volts have been in use for several years and tests have been run on 1,000,000-volt experimental systems.

High-tension transmission makes possible the use of large central stations where generating units of large capacities can be used. Alternating-current generators having efficiencies of 95 per cent or better, capacities of 200,000 kva, and speeds of 1800 rpm are now in service, whereas it is difficult to build d-c generators of over 1000 kw at speeds above 1000 rpm. In steam power plants a-c generators are usually driven by steam turbines at 3600 or 1800 rpm, depending on the size of the unit. In order to obtain high efficiency the turbine must operate at high speeds, hence reduction gears are commonly used when d-c generators are turbine driven. This means greater initial costs and lower operating efficiencies for turbine driven d-c units.

Induction motors which operate on polyphase (multiple voltage) a-c systems are simple in design, and for equal sizes and speeds cost about one-half as much as d-c motors. They require very little maintenance when compared to d-c motors.

There are some applications where direct current has advantages. The multiple-phase induction motor is inherently a constant-speed motor, and although a-c adjustable-speed motors are available, they are, generally speaking, not as satisfactory as d-c motors for drives which require adjustable speed. Furthermore, a-c adjustable-speed motors usually are more costly than d-c adjustable-speed motors. Motors operating on 2-wire a-c systems are usually not as satisfactory as d-c motors for applications requiring sizes larger than 5 hp. The field windings of a-c generators and synchronous motors must be excited with direct current. Small d-c generators used for this purpose are called exciters, and are often mounted on the shaft of the a-c machine. Direct current is used extensively in telephony, radio communication, and signaling. Wherever a battery of any type is required, there is an application for direct current.

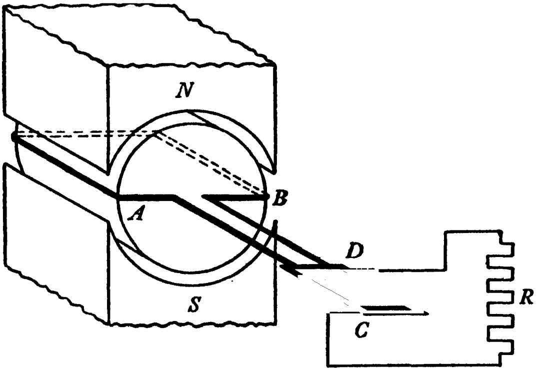


Fig. 1-1. Elementary Alternating-Current Generator.

Much of the direct current used for various purposes is originally generated as alternating current. The alternating current is transmitted to the vicinity of the load and converted into direct current by motor-generator sets, synchronous converters, or rectifiers.

2. Production of Alternating Voltage and Current. Fig. 1-1 shows an elementary alternating-current generator consisting of a one-turn coil *AB* which is free to rotate in a magnetic field. End *A* of the coil is connected to slip-ring *C* and end *B* terminates in slip-ring *D*. Assume the coil to rotate counterclockwise from the position as shown in Fig. 1-1. At the instant rotation is begun, conductors *A* and *B* move parallel to, but do not cut across, the magnetic field. Hence, no voltage is induced at this instant. As the coil continues to rotate, conductors *A* and *B* cut through the field, and a voltage is induced in the coil. If the right-hand generator rule is applied, it will be seen that the direction of this voltage is from A to B during the first 180 degrees of rotation. As the coil reaches the 180-degree position, the conductors again move parallel to the flux, and the voltage drops to zero. Between 180 and 360 degrees conductors A and B move across the magnetic field for a second time, but in opposite directions to their motion during the first 180 degrees. The voltage is now from B to A, or opposite to the direction of voltage induced during the first 180 degrees. For each revolution of the coil, the voltage passes from zero to a maximum value in one direction, decreases through zero to a maximum value in the opposite direction, and returns to zero. A voltage of this type is known as an *alternating* voltage.

If the 360 degrees of rotation are laid out along a horizontal axis, and the instantaneous values of voltage generated at various angular positions of the coil are plotted vertically, a wave-shaped curve similar to that of Fig. 1-2 probably would be obtained. The actual shape of the alternating voltage wave induced in a single coil is determined by the pattern of flux distribution in the air gap. In commercial alternators several coils are connected in series and the alternating voltage wave obtained from the series group more nearly approximates a *sine wave* as shown in Fig. 1-3.

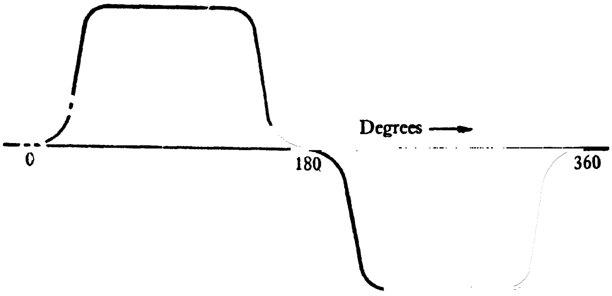


Fig. 1-2. Voltage Wave of Single Coil Alternator.

When the direction of voltage in the coil (Fig. 1-1) is from A to B, the current flows through the resistor R from D to C, and when the voltage reverses, the current also reverses and flows from C to D. The magnitude of the current through R rises and falls with the magnitude of the voltage. Thus, the wave form of the current in the circuit illustrated follows the same pattern as the voltage wave, and the current is said to be alternating. It is not correct to assume that the current wave in all a-c circuits passes through zero and other corresponding values in step with the voltage wave, for such is not always the case. This topic will be discussed in more detail later in the chapter. Neither is it correct to assume that the current wave takes the same shape as the voltage wave under all circuit conditions. However, for the present we are interested in the more general case in which current and voltage waves are so nearly of sine-wave form that they may be treated mathematically as true sine waves.

3. Definitions. The voltage wave generated by a conductor completing the passage of one north and one south magnetic pole is called a *cycle.* The coil (Fig. 1-1) produces one cycle of voltage per revolution. During one-half cycle the wave passes through one *alternation.* The number of cycles completed in one second is the *frequency* of the wave. Frequency is expressed in *cycles per second* (*cps*) and is denoted by the symbol f. The time in seconds required for the wave to pass through one cycle is its *period.* A wave having a frequency of 60 cycles per second requires 1/60 second to pass through one cycle; hence, the period of a 60-cycle wave is 1/60 second.

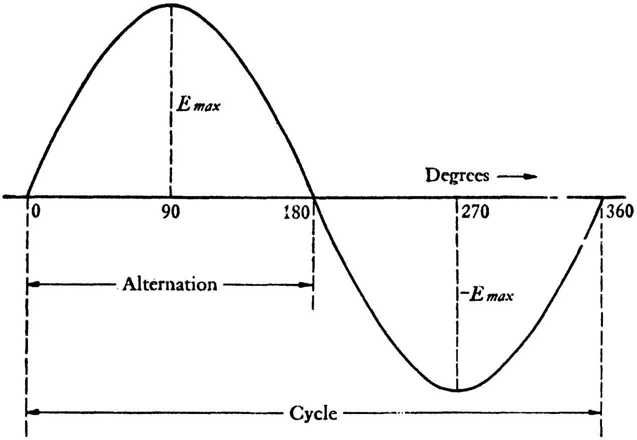


Fig. 1-3. Sine Wave of Voltage.

4. Relation between Time and Space-Degrees. In the sine wave (Fig. 1-3) the magnitude of voltage is represented along the vertical axis and the space-degrees through which the conductors pass are plotted horizontally. Since the conductors which generate this wave move with uniform angular velocity, there is a definite relationship between time and space-degrees; and the wave could have been constructed with time instead of space-degrees along the horizontal axis. Thus, for a 60-cycle wave the 360 degrees would correspond to a time of 1/60 second. Other corresponding values between degrees and time for a 60-cycle wave are illustrated in Fig. 1-4.

5. Electrical Time-Degrees and Space-Degrees. One cycle is produced in a 2-pole alternator when the conductors pass through 360 space-degrees. The time required for the cycle is $1/f$ second, where f represents the frequency of the wave. The time $1/f$ second may be divided into 360 equal divisions, each called an *electrical time-degree.* Then, in a 2-pole alternator one space-degree equals one electrical time-degree. In a 4-pole alternator, the wave passes through two cycles, or 720 electrical time-degrees, while the conductors rotate through 360 space-degrees. Thus, in the case of a 4-pole alternator one mechanical space-degree is the equivalent of two electrical time-degrees. It is obvious that the relationship between electrical time-degrees and space-degrees depends on the number of poles in the alternator. If alternating voltage and current waves were plotted with space-degrees, the number of degrees per cycle would vary

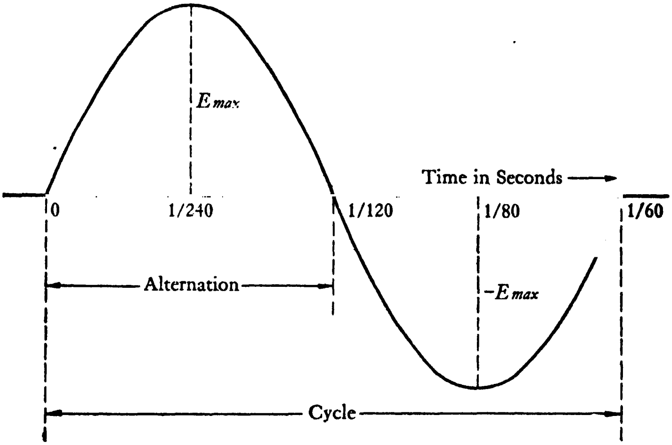


Fig. 1-4. A 60-Cycle Sine Wave with Time in Seconds Plotted as the Abscissa.

with the number of poles in the alternator. Since the number of electrical time-degrees per cycle is 360 regardless of the number of poles, it is customary to plot alternating voltage and current waves with electrical time-degrees along the horizontal axis.

Problem 1-1: The period of a sine wave is 0.04 second. Determine: (*a*) the frequency; (*b*) the time required for the wave to pass from zero to a positive maximum; (*c*) the time for each electrical time-degree.

Problem 1-2: A sine wave has a frequency of f cps. Express in terms of f: (*a*) the time for one alternation; (*b*) the time for one electrical time-degree.

Problem 1-3: In an 8-pole alternator one space-degree equals how many electrical time-degrees?

Problem 1-4: Through how many electrical time-degrees does a conductor for a 12-pole alternator pass while rotating through an arc of 45 space-degrees?

6. Relation between Number of Poles, RPM and Frequency of an Alternator. One cycle has been defined as that portion of a wave produced by armature conductors completing the passage of two poles. Hence, the number of poles and the rpm of the conductors determine the frequency of an alternator. If we let P represent the number of poles, the number of cycles per revolution equals $P/2$. And, if s represents the revolutions per second,

$$f = Ps/2 \text{ cps}$$

Allowing S to represent the revolutions per minute,

$$s = S/60$$

Then:

$$f = PS/120 \text{ cps} \tag{1-1}$$

Problem 1-5: An alternator with 10 poles revolves at 600 rpm. What is its frequency?

Problem 1-6: At what rpm must a 6-pole alternator be driven to produce a frequency of 25 cps?

Problem 1-7: An alternator driven at 150 rpm must have how many poles in order to produce 60 cps?

Problem 1-8: Determine the frequency of a 10-pole alternator which is driven at 480 rpm.

Problem 1-9: A 4-pole alternator is to be driven at varying speeds to provide frequencies ranging from 35 to 70 cps. Determine its minimum and maximum rpm.

7. Standard Commercial Frequencies. The two standard commercial frequencies in the United States are 25 cps and 60 cps. The voltage drop per unit of current and distance is less with the lower frequency and hence, the voltage regulation on transmission lines is better at 25 cps than at 60 cps. Motors and synchronous converters function better at the lower frequency, although this advantage is not significant except for certain types of machines. The flicker of incandescent lamps at 25 cps is objectionable and for this reason the higher frequency is desirable for electric systems which supply energy for both power and lighting. The transformers used on lower frequency systems are heavier and more costly.

8. Sine Waves. One of the chief advantages of alternating voltage is the ease with which it may be transformed. Of all wave forms, sine waves can be transformed with the least distortion. For this reason designers of alternators look upon sine-wave shape as ideal and attempt to follow it in their designs as closely as practicable. The emfs of commercial alternators are seldom perfect sine waves, but usually are close enough to sine form to permit treating them mathematically as sine waves.

In producing one cycle of alternating voltage, a conductor passes through 360 electrical time-degrees in $1/f$ sec. If we convert the electrical time-degrees into electrical time-radians, 360 degrees would equal 2π radians. The electrical angular velocity ω (omega) would be

$$\omega = \frac{2\pi}{1/f} = 2\pi f \text{ radians per second} \tag{1-2}$$

The electrical time angle in radians corresponding to a time t seconds would be $2\pi ft$ or ωt radians.

If the magnitude of voltage for each value of t equals the maximum value times the sine of the angle passed through in t seconds, the voltage is of sine-wave form. The equation of this wave is

$$e = E_{max} \sin \omega t \tag{1-3}$$

where e is the instantaneous voltage at the angle ωt and E_{max} is the maximum voltage.

Likewise, the equation for a sine-wave current is

$$i = I_{max} \sin \omega t \tag{1-4}$$

Example 1-1: A sine wave of emf has a frequency of 40 cycles per second and a maximum value of 325 volts. Determine: (*a*) the angular velocity of the wave; (*b*) the instantaneous value of voltage when ωt equals 0.45 radian; and (*c*) the instantaneous value of voltage when ωt equals 30°.

Solution.

(*a*) $\omega = 2\pi f = 6.28 \times 40 = 251.2$ radians per second *Ans.*

(*b*) $e = E_{max} \sin \omega t = 325 \times 0.435 = 141.38$ volts *Ans.*

(*c*) $e = E_{max} \sin \omega t = 325 \times 0.5 = 162.5$ volts. *Ans.*

Example 1-2: How long in seconds is required for a sine wave having a frequency of 25 cps to pass from its zero value to one-half its maximum value?

Solution.

$$\begin{aligned} e &= E_{max} \sin \omega t \\ \omega t &= \sin^{-1} e/E_{max} = \sin^{-1} 0.5 \\ \omega t &= 30^\circ \\ t \text{ per cycle} &= 1/25 = 0.04 \text{ sec} \\ t \text{ per degree} &= 0.04/360 = 0.000111 \text{ sec} \\ t \text{ for } 30^\circ &= 30 \times 0.000111 = 0.00333 \text{ sec.} \end{aligned}$$ *Ans.*

(The expression $\sin^{-1}$ is to be read as "*the angle whose sine is.*")

Problem 1-10: (*a*) What is the instantaneous value of voltage at 45° when the maximum value is 140 volts? (*b*) What is the instantaneous value of the voltage at 225°?

Problem 1-11: If the instantaneous value of a sine wave of current is 36 amp at 135 degrees, what is the maximum value of the wave?

Problem 1-12: The instantaneous value of a sine wave of voltage is 87 volts at 60°. (*a*) What is its instantaneous value at 30°? (*b*) at 150°? (*c*) at 180°? (*d*) at 210°? (*e*) at 330°?

Problem 1-13: (*a*) What is the angular velocity of a 60-cycle sine wave expressed in radians per second? (*b*) Through what angle, expressed in radians, does a 60-cycle sine wave pass in 1/480 of a second? (*c*) What is the instantaneous value of a

60-cycle sine wave 1/480 of a second after it has passed through 0°, if its maximum value is 150?

Problem 1-14: *(a) What is the frequency of a sine wave which has an angular velocity of 314 radians per second? (b) How long in seconds is required for this wave to pass from zero to 0.707 times its maximum value?*

9. Construction of a Sine Wave. The circle (Fig. 1-5) is drawn to a convenient scale with the radius equal to the maximum voltage. The circumference of the circle is divided into a number of equal arcs. In Fig. 1-5 twelve arcs, each spanning an angle of 30°, have been used. A horizontal line *CD* is divided into the same number of equal parts as there are divisions of the circle. Each division of *CD* represents the time required for the coil to move through 30°, and the entire length of *CD* represents the time required for one complete revolution. The point where the radius at 30° cuts the circumference of the circle is projected horizontally until it intersects a perpendicular to *CD* at the 30° division. The length of the perpendicular from *CD* to the point of intersection represents to scale the instantaneous value of emf at 30°. That is

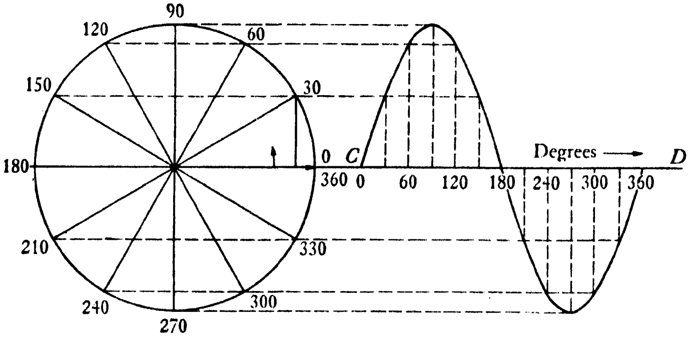


Fig. 1-5. Construction of a Sine Wave.

$$\sin 30° = e/E_{max} \quad \text{or} \quad e = E_{max} \sin 30°$$

Horizontal lines projected from other intersections on the circumference of the circle to perpendiculars at corresponding angles on *CD* provide points for construction of the sine wave. The curve for angles between 0° and 180° lies above line *CD* representing positive values of emfs, and for angles between 180° and 360° the curve represents negative values of emfs.

Problem 1-15: Construct a sine wave of current which has a maximum value of 25 amp. *Suggestion:* Use a piece of $8\frac{1}{2}$ by 11 inch cross-section paper and place the long side horizontally. With a scale of one inch equal to 10 amp draw a semicircle with radius equivalent to the maximum current. Center of semicircle should be on left grid $3\frac{1}{2}$ inches from bottom grid. Divide the semicircle into 12 angles of 15°

each. Lay out 360° on the horizontal time axis with a scale of one-quarter inch equal to 15°. Obviously, it is not necessary to draw the complete circle.

Problem 1-16: *Construct a sine wave of emf which has an instantaneous value of 35.35 volts at 135°. Suggestion: Same as for problem 1-15 except use scale of one inch equal to 20 volts.*

10. Average Height of One-Half Cycle of Sinusoidal Voltage or Current. The average height, or average ordinate, of the sine wave is an important factor in the study of alternating current and voltage waves. As the shape of the positive and negative alternations of a sine wave is the same, the average ordinate of the wave for a complete cycle is zero. The average ordinate for one alternation, however, has a definite value and can be found by measuring the area under one alternation and dividing this area by the length of the alternation along the time axis. If the work is done accurately, the ratio will be 0.636. Thus for a sine wave of current:

$$I_{ave} = 0.636 I_{max} \tag{1-5}$$

Likewise for a sine wave of emf:

$$E_{ave} = 0.636 E_{max} \tag{1-6}$$

The area under one alternation of a sine wave can be measured with a planimeter (an instrument for measuring area), or if the wave is plotted on cross-section paper, the area can be estimated by counting the small squares which lie under the curve. Also, the area under one alternation can be computed mathematically by use of the calculus. By computation the area equals twice the maximum height. Dividing this value by the length, π radians, we have:

$$I_{ave} = 2I_{max}/\pi = 0.636 I_{max}$$

Likewise:

$$E_{ave} = 2E_{max}/\pi = 0.636 E_{max}$$

The average or mean ordinate of a sine wave can be estimated without determining the area under the curve by adding a number of ordinates which are equally spaced from 0° to 180° and dividing this sum by the number of ordinates. Obviously, the accuracy of determining the mean ordinate by this method increases with the number of ordinates selected.

Problem 1-17: (*a*) Determine the average value of the sine wave of current, problem 1-15, by measuring the area of one loop of the wave and dividing the area obtained by the length of the loop along the time axis. (*b*) Determine the ratio of the average ordinate to the maximum ordinate of the wave.

Problem 1-18: (*a*) Determine the average value of the sine wave of emf, problem 1-16, by measuring the area of one loop of the wave and dividing the area obtained by the length of the loop along the time axis. (*b*) Determine the ratio of the average ordinate to the maximum ordinate of the wave.

Problem 1-19: If the average value of a sine wave of emf is 96 volts, what is its maximum value?

Problem 1-20: The instantaneous value of a sine wave of current at 38° is 27.4 amp. What is its average value?

11. The Alternating-Current Ampere. The ampere of direct current is defined as *that value of current which will deposit 0.001118 grams of silver per second in a standard solution of nitrate of silver and water.* The ampere of alternating current cannot be defined in this manner as the silver deposited in one-half cycle would be removed during the other half. The ampere of alternating current is defined as *that value of alternating current which will produce the same heating effect as one ampere of direct current when flowing through the same ohmic resistance for the same period of time.* This value of alternating current is called the *effective value* and is the value read by alternating-current ammeters of the more generally used types.

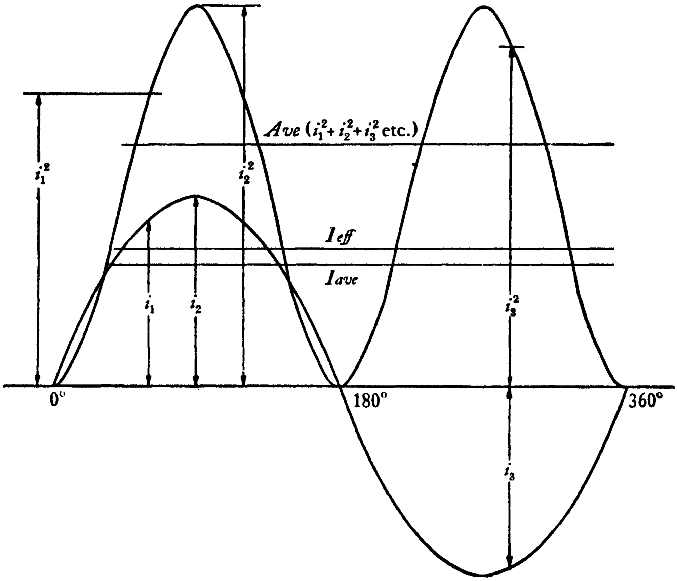


Fig. 1-6. Effective Value of Sine-Wave Current.

12. Effective Value of Alternating Voltage or Current. In an a-c circuit the instantaneous power equals the product of the resistance and the square of the instantaneous current. That is, $P_1 = Ri_1^2$, $P_2 = Ri_2^2$, $P_3 = Ri_3^2$, etc. The average power is:

$$P_{ave} = \text{average of } (Ri_1^2 + Ri_2^2 + Ri_3^2, \text{ etc.})$$
$$P_{ave} = R\ (\text{average of } [i_1^2 + i_2^2 + i_3^2, \text{ etc.}])$$

Also,

$$P_{ave} = RI_{eff}^2$$

Therefore,

$$I_{eff}^2 = (\text{average of } [i_1^2 + i_2^2 + i_3^2, \text{etc.}])$$

And,

$$I_{eff} = \sqrt{\text{average of } (i_1^2 + i_2^2 + i_3^2, \text{etc.})}$$

Do not confuse I_{ave}^2 with the average of $(i_1^2 + i_2^2 + i_3^2, \text{etc.})$, for the two values are not the same and, therefore, $\sqrt{\text{average of } (i_1^2 + i_2^2 + i_3^2, \text{etc.})}$ or I_{eff} is not the same value as I_{ave}.

In Fig. 1-6 the squares of the instantaneous values of current are used to produce the squared-current curve. If the area under the squared-current curve is measured and divided by the length, the average ordinate of the squared-current curve will be obtained. The effective value of current, often called the *root mean square* (*rms*) value, equals the square root of this average or mean ordinate of the squared-current curve. If the effective value of current is divided by the maximum value, the ratio will be 0.707. That is:

$$I_{eff} = 0.707 I_{max} \tag{1-7}$$

The relationship between the effective value of voltage (rms voltage) and the maximum value of the voltage wave is the same as that between rms current and the maximum value of the current wave or:

$$E_{eff} = 0.707 E_{max} \tag{1-8}$$

When we speak of an a-c current or voltage, the effective value is understood unless otherwise stated. Subscripts are used to denote average or maximum values such as E_{ave} or I_{max}, but it is customary to omit subscript when writing effective values. Thus the symbols E and I without subscripts indicate effective values.

Attention is called to the fact that equations 1-7 and 1-8 are derived with the assumption that the wave form of current or voltage is sinusoidal. Hence, these equations will be in error when wave forms other than sinusoidal form are encountered.

Problem 1-21: Place the long side of an $8\frac{1}{2}$ by 11 inch sheet of cross-section paper horizontally. Construct a sine wave of current having a maximum value of 2 amp. Locate the center of the semicircle on the left grid of the paper 2 inches above the lower left corner. Use scale of 1 inch = 1 amp and 1/4 inch = 15°. Square the instantaneous current ordinates at 15° intervals, and plot the squared-current curve. Determine the average ordinate of the squared-current curve by measuring its area and dividing by its length. One or both loops of the squared-current curve may be used. Take the square root of this average ordinate to find the value of I_{eff}. Divide the effective value of current as found by the maximum value to obtain the ratio by which the maximum must be multiplied in order to obtain the effective value. If the work is accurate, this ratio will be 0.707.

Problem 1-22: What is the effective value of an alternating current of sine wave form which has a maximum value of 40 amp?

Problem 1-23: The effective value of a-c voltage used in residence lighting is 115 volts. What is the maximum voltage?

13. Effective Resistance. A coil (Fig. 1-7) with a removable iron core is connected across a d-c source. Resistance in series with the coil is gradually decreased until the current rises to 1.2 amp, a safe value for the coil. The voltmeter reading is 20 volts and the wattmeter reading is 24 watts. The conductor or ohmic resistance, $R_{d\text{-}c}$, calculated from voltage and current is:

$$R_{d\text{-}c} = E/I = 20/1.2 = 16.67 \text{ ohms}$$

Or calculated from power and current:

$$R_{d\text{-}c} = P/I^2 = 24/1.2^2 = 16.67 \text{ ohms}$$

Regardless of the position of the iron core the d-c instruments give the same indication. These, however, will vary while the core is in motion.

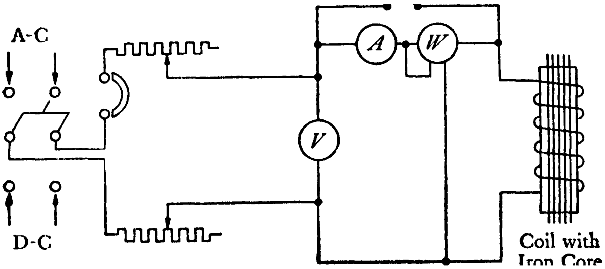


Fig. 1-7. Circuit for Illustrating Ohmic and Effective Resistance.

The iron core is removed from the coil, the d-c voltmeter and ammeter are replaced with a-c instruments, and a 60-cycle a-c source is impressed on the circuit. Resistance in series with the coil is decreased until the current reaches the same value as obtained with dc, i.e., 1.2 amp. It now requires 48 volts to establish this value of current in the coil, and the wattmeter reads 24 watts. The magnetic field set up in the coil by the alternating current is changing continuously, and, thus a voltage of self-induction is generated in the coil. According to Lenz's law this induced voltage opposes the applied voltage. Thus, the applied voltage has two components: one which overcomes the voltage of self-induction, and the other which overcomes the resistance drop. This accounts for the increase from 20 to 48 volts.

The power in the circuit with alternating voltage applied is less than the product of E and I, and hence the *volt-amperes* (product of volts and amperes) must be multiplied by a factor less than unity in order to equal the watts indicated by the wattmeter. This coefficient of EI is known as the *power factor* of the circuit. Since the full value of applied voltage is not spent in overcoming the resistance drop of the circuit, E divided by I would not be R, but would be the combined effect of resistance and the voltage of self-induction to impede the flow of current. The value E divided by I gives the *impedance* of the circuit. The unit of impedance is the *ohm* and the symbol for impedance is Z.

$$Z = E/I \tag{1-9}$$

The conductor resistance with alternating voltage applied is determined by dividing the power by the square of the current.

$$R_{a\text{-}c} = P/I^2 = 24/1.2^2 = 16.67 \text{ ohms}$$

The conductor resistance with the iron core removed apparently remains constant regardless of whether direct or alternating voltage is applied. This, however, is not actually true, for the changing magnetic field set up by the alternating current induces a voltage which tends to have greater magnitude near the center of the conductor. The effect of this induced voltage is to set up eddy currents in the conductor, and to cause the current to be denser near the surface. This latter phenomenon is known as *skin effect*. The difference between d-c and a-c conductor resistance increases with the cross-section area of the conductor and the frequency, and is of little importance, practically speaking, until the product of the cross-section area in circular mils and the frequency exceeds 10,000,000.

A third set of readings is taken with the iron core in the coil and a-c voltage applied. The series resistance is again adjusted to give 1.2 amp and the voltage impressed on the coil is now 120 volts. The wattmeter reading is 33.7 watts. Placing the iron core in the coil reduced the reluctance of the magnetic circuit and thus increased the magnetic flux. Hence, the voltage of self-induction was increased, and a greater value of applied voltage was required to establish 1.2 amp in the circuit. The wattage increased due to hysteresis and eddy current losses in the iron core, and hence, the wattage divided by the square of the current gives a higher value of resistance. This value of resistance is known as the *effective resistance* of the circuit, and in the case stated:

$$R_{eff} = P/I^2 = 33.7/1.2^2 = 23.4 \text{ ohms}$$

The entire electrical input to the coil (I^2R) is radiated as heat. Part of this heat is generated in the copper wire with which the coil is wound, and part is generated in the iron core. Since the eddy currents and hysteresis produce the same effect in the circuit as resistance, i.e., heat, their effect in the a-c circuit is included as part of the effective resistance.

If a core which is nonmagnetic, but which is a conductor of electricity, were inserted in the coil in place of the iron core, the eddy currents induced by the a-c flux would cause the wattage and effective resistance to be greater at the same current value than for the coil with the core removed. We may conclude then, that ohmic resistance is the actual resistance of the conductor as measured with direct current, while effective resistance takes into consideration, not only ohmic resistance, but skin effect and core losses of the circuit as well. *Do not confuse effective resistance with impedance.*

14. A-c Circuits Containing Only Resistance. Fig. 1-8 shows one of several methods used to construct resistors for service in a-c circuits. It will be noted that by winding the coil *two-in-hand* the magnetomotive force developed by one-half of the turns would be cancelled by the remaining turns. Except for currents of very high frequencies such a coil is noninductive and its windings offer only ohmic resistance to the passage of current.

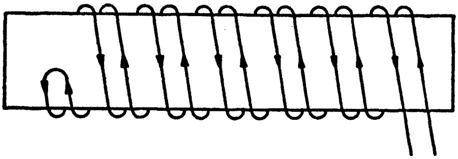


Fig. 1-8. Noninductive Resistor.

When a noninductive resistor is connected across an a-c source, the wattage equals the product of the applied voltage and the current. That is:

$$P = EI \qquad (1\text{-}10)$$

Hence, *in a circuit containing only resistance, the power factor is unity*, and the resistance may be found, either by dividing the voltage by the current, or by dividing the power by the square of the current.

$$R = E/I = P/I^2 \qquad (1\text{-}11)$$

Fig. 1-9 shows the voltage and current waves for an a-c circuit containing only noninductive resistance. The two waves pass through zero,

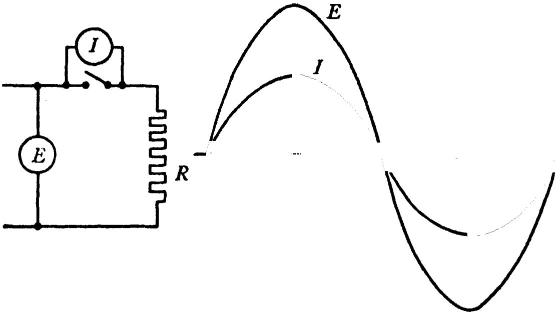


Fig. 1-9. Phase Relation Between Current and Voltage in an A-c Circuit Containing Only Resistance.

positive maximum, and other corresponding ordinates at the same instants, i.e., there is no time interval, angular displacement, or *phase angle,* between the waves. Under this condition the waves are said to be *in phase.*

Problem 1-24: A circuit draws 12 amp and 720 watts when connected across a 120-volt a-c source. Determine the impedance and the effective resistance.

Problem 1-25: The high-voltage coil of a transformer is connected across a 220-volt a-c source with the low-voltage coil open-circuited. The coil draws 1.4 amp and 80 watts. Determine its impedance and effective resistance.

15. Voltage of Self-Induction.

A change of current in a circuit causes a corresponding change of the magnetic field. This change in magnetic field in turn induces a voltage of self-induction in the circuit. The *average value* of this self-induced voltage:

$$E_{si} = -N\ (\phi\ change)/10^8 t \text{ volts} \qquad (1\text{-}12)$$

where N is the number of turns, ϕ is the flux, and t is the time in seconds during the change in flux.

Self-inductance of a circuit is the rate of flux linkage change in respect to the rate of current change. When the magnetization curve of the circuit is a straight line, as in an air core coil, the inductance is constant and can be computed from the equation:

$$L = N\ (\phi_2 - \phi_1)/10^8\ (I_2 - I_1) \text{ henry} \qquad (1\text{-}13)$$

where $(\phi_2 - \phi_1)$ represents the change in flux and $(I_2 - I_1)$ is the change in current. When magnetic material is present in the core of a coil the magnetization curve is not linear, and L is not constant. Under this condition the *average value of inductance* between the limits of current is obtained when equation 1-13 is used. Rearranging equation 1-13 to read:

$$L\ (I\ change) = N\ (\phi\ change)/10^8$$

and substituting in equation 1-12, we have

$$E_{si} = -L\ (I\ change)/t \text{ volts} \qquad (1\text{-}14)$$

16. Angular Displacement Between *E* and *I* in an A-c Circuit Containing Only Inductance.

Fig. 1–10 shows the voltage, current, and flux waves for an a-c circuit containing only inductance. It will be noted that the rate of change of the current and flux waves is greatest when they are passing through zero. Hence, E_{si} will pass through its maximum value when current and flux pass through their zero values. Also, the rate of change of current and flux is zero as the waves pass through their maximum values. Hence, E_{si} will be zero when I and ϕ are maximum. Thus, an angular displacement of 90° exists between the current wave and the voltage of self-induction wave.

According to Lenz's law, E_{si} will be negative while I is increasing from negative maximum to positive maximum, and positive as the current decreases from positive maximum to negative maximum. Since the circuit contains only inductance, E_{si} is equal and opposite to the applied voltage, E. Hence, *in an a-c circuit which contains only inductance, a phase angle of 90° exists between the applied voltage and the current.* In this circuit I passes through its zero, positive maximum, and other corresponding values 90° later than E. Thus, *I lags E by 90°*, and the phase angle is an angle of lag. Phase angle is denoted by the symbol θ, and should not be confused with the time angle ωt.

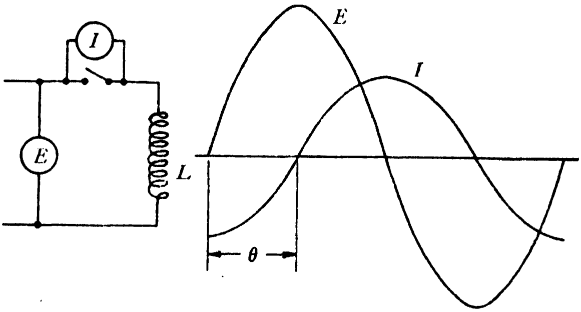


Fig. 1-10. Phase Relation Between Current and Voltage in an A-c Circuit Containing Only Inductance.

Example 1-3: A 25-cycle current having an effective value of 5 amp flows in a circuit having an inductance of 0.2 henry. What is the average voltage of self-induction generated while the current is passing from its zero value to its instantaneous value at 10°?

Solution.

$$I_{max} = 5/0.707 = 7.07 \text{ amp}$$

$$i_{10} = I_{max} \sin 10^\circ = 7.07 \times 0.174 = 1.23 \text{ amp}$$

$$I \text{ change} = i_{10} - i_0 = 1.23 \text{ amp}$$

$$t = 1/25 \times 10/360 = 10/9000 = 1/900 \text{ sec}$$

$$E_{si} = \frac{-L\,(I \text{ change})}{t} = -\left(\frac{0.2 \times 1.23}{1/900}\right)$$

$$= -\,(900 \times 0.2 \times 1.23) = -\,221.4 \text{ volts} \qquad \textit{Ans.}$$

The voltage calculated in the above solution is the average of the self-induced voltage while the current is passing from zero to 10° and should not be confused with the average for the entire change between zero and I_{max}. The value is negative because the current is increasing and E_{si} opposes an increase.

Problem 1-26: A coil of negligible resistance has an inductance of 0.318 henry. A 25-cycle current having an effective value of 2.2 amp flows through the coil. Plot to scale the current wave, the voltage of self-induction wave, and the wave of applied voltage. *Suggestion:* Place a sheet of $8\frac{1}{2}$ by 11-inch cross-section paper with the long side in the horizontal position. Draw the time axis horizontally across the sheet $3\frac{1}{2}$ inches from the bottom grid. Construct 180° of the current wave using a scale of 1 inch = 1 amp and 1 inch = 20°. Calculate the average voltage of self-induction for each 10° of the current wave, i.e., between 0° and 10°, between 10° and 20°, etc. Plot the average voltage of self-induction half-way between the angular limits for which it is obtained. Use a scale for E and E_{si} of 1 inch = 50 volts. Remember that E_{si} is negative when I is increasing and positive when I decreases. E is plotted equal and opposite in sign to E_{si}.

17. Determining the Effective Value of E_{si} in an A-c Circuit.

The current in an a-c circuit changes from zero to I_{max} in one-quarter of a cycle, i.e., in $1/4f$ cycles. If the inductance of the circuit is L, the flux-linkage change as the current passes from zero to maximum is

$$L \times 10^8 \times I_{max}$$

The average voltage of self-induction during this change,

$$E_{ave} = -\frac{L \times 10^8 \times I_{max}}{10^8 \times 1/4f} = -4fLI_{max}$$

$$E_{max} = \frac{E_{ave}}{0.636} = -\frac{4fLI_{max}}{0.636} = -6.28fLI_{max}$$

Multiplying both sides of the equation by 0.707 to obtain effective values,

$$(0.707E_{max}) = -6.28fL\,(0.707I_{max})$$

$$E_{si} = -6.28fLI = -2\pi fLI \qquad (1\text{-}15)$$

The component of the applied voltage which overcomes E_{si} is denoted by the symbol E_L. Hence, E_L is equal and opposite to E_{si} and can be expressed as:

$$E_L = 2\pi fLI \qquad (1\text{-}16)$$

Example 1-4: A 50-cycle source is impressed across a coil having an inductance of 0.4 henry, and 3 amp flows in the coil. Determine the component of the applied voltage which overcomes E_{si}.

Solution.

$$E_L = 2\pi fLI$$

$$E_L = 6.28 \times 50 \times 0.4 \times 3 = 376.8 \text{ volts} \qquad \textit{Ans.}$$

18. Inductive Reactance.

Because the voltage of self-induction opposes the change in current at any instant, its effect on the circuit is measured in ohms. This tendency of E_{si} to react against the change in current is called *inductive reactance,* and its symbol is X_L. Inductive *reactance equals the component of the applied voltage which overcomes E_{si}, divided by the current.*

$$X_L = E_L/I \tag{1-17}$$

Substituting for E_L its value from equation (1-16), we have

$$X_L = 2\pi fLI/I = 2\pi fL = \omega L \tag{1-18}$$

Since equation 1-16 is derived assuming ratios between average, effective and maximum values as obtained from sine waves, it is evident that equation 1-18 is true only when the wave follows the sine law.

Example 1-5: A coil of negligible resistance having an inductance of 0.5 henry is placed across a 60-cycle, 220-volt supply. What current flows in the coil?

Solution.

$$X_L = 2\pi fL = 6.28 \times 60 \times 0.5 = 188 \text{ ohms}$$
$$I = E/X_L = 220/188 = 1.17 \text{ amp} \qquad \textit{Ans.}$$

Problem 1-27: A coil having an inductance of 0.03 henry is placed across a 60-cycle source. A current of 5 amp flows in the circuit. Determine the component of the applied voltage required to overcome E_{si}.

Problem 1-28: A coil of 450 turns has an effective voltage of 120 volts induced in its windings when a 60-cycle current of 4 amp flows in the circuit. (*a*) What is the inductance of the coil? (*b*) What is the maximum flux linkage? (*c*) What is the maximum flux change?

Problem 1-29: A coil of negligible resistance draws 6 amp when placed across a 120-volt, 25-cycle source. (*a*) What is the inductance of the coil? (*b*) What is the maximum flux linkage for one-fourth of a cycle?

Problem 1-30: A transformer primary draws 2 amp when placed across a 240-volt, 60-cycle source. What current will the winding draw if placed across a 25-cycle, 240-volt source? (Consider the inductance of the coil to be constant at both frequencies and resistance of the coil to be negligible.)

Problem 1-31: If the inductance L of the transformer primary, problem 1-30, is reduced 30 per cent due to saturation of the core when the frequency is changed from 60 to 25 cycles, what current will it draw when placed across the 25-cycle, 240-volt a-c source?

19. Analogy of an A-c Circuit Containing Only Capacitance. The phase relation between current and voltage in an a-c circuit containing only capacitance is analogous to the relation between flow and pressure in the hydraulic system (Fig. 1-11). Pressure is produced in the pipe by a paddle-type pump which may be driven with varying speed in either direction. A flexible rubber diaphragm extends across the enlarged section of the pipe. Consider the pressure to be positive when it tends to force the liquid in a counterclockwise direction and to be negative when it tends to send it clockwise. Also, consider the current to be positive when in the counterclockwise direction and to be negative when clockwise.

With equal pressure on each side of the diaphragm, the pump is started in a counterclockwise direction and its speed is regulated so as to give sine wave pressure variation with respect to time, as shown by Fig. 1-12. Current flows counterclockwise or positive while the pressure is increasing, and establishes a back pressure by stretching the rubber diaphragm to the left. Thus, flow of the liquid is the means by which the applied pressure and back pressure tend to counterbalance each other. During the period in which the pressure of the pump increases from zero to maximum in a counterclockwise direction, current flows counterclockwise. The magnitude of the current at any instant is proportional to the rate at which the pump pressure is changing at that instant. And since we have assumed sine-wave form for the pressure of the pump, the rate of change of the pump pressure will be a maximum at the instant the wave passes through zero. Hence, the current will be a maximum positive value while the pump pressure is increasing in a positive direction through zero. At the instant the pressure of the pump reaches its maximum positive value, its rate of change will be zero and current will cease to flow in the system.

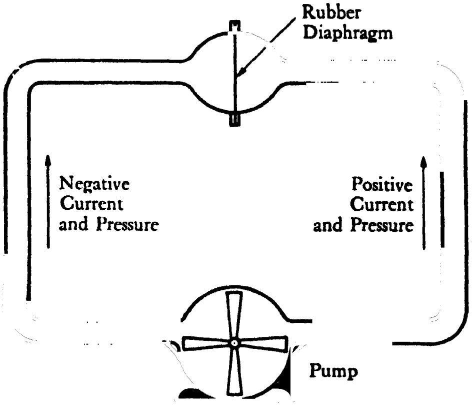


Fig. 1-11. Hydraulic Analogy of Capacitance in an A-c Circuit.

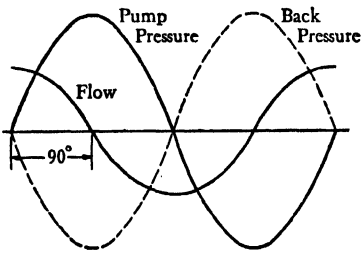


Fig. 1-12. Phase Relation Between Flow and Pump Pressure for the Hydraulic Circuit of Fig. 1-11.

It is not necessary to reverse the pump pressure to cause the current to reverse. Simply allow the pump pressure to decrease from its maximum positive value and the back pressure of the diaphragm sends the current in the clockwise direction. The current continues to flow clockwise and increases to its maximum negative value at the instant the pump pressure reaches zero. At this instant the rubber diaphragm is in its original unstretched position and the back pressure is zero.

Reversing the pump produces pressure in a clockwise direction and current continues to flow clockwise. The magnitude of the current increases to zero as the pump pressure continues to decrease to a negative maximum. The continued motion of current in the clockwise direction causes the rubber diaphragm to stretch to the right and when the pump pressure reaches its negative maximum, the back pressure reaches its positive maximum. During the final quarter-cycle the pump pressure, which is still clockwise, decreases from a maximum in this direction to zero. The back pressure of the diaphragm causes current to flow counterclockwise, the magnitude of which increases from zero to maximum. Hence, in this hydraulic circuit the current leads the applied pressure by an angle of 90°.

20. Current and Voltage Relations in an A-c Circuit Containing Only Capacitance. Consider the sine-wave voltage E (Fig. 1-13) to be positive when it acts counterclockwise, and negative when it acts clockwise. Likewise, consider the current to be positive when counterclockwise and negative when clockwise. During the first quarter-cycle while the applied voltage is increasing in a positive direction from zero to a maximum positive value, the current flows from the left plate of the capacitor, through the generator, to the right plate. Voltage and current are both positive. Since the magnitude of current at any instant is proportional to the rate of change of voltage at that instant, the current will be at its maximum positive value when the voltage is passing through zero in a positive direction. Also, the current will be zero when the applied voltage is at its maximum positive value. The flow of current in the positive direction during this period of time charges the right plate of the capacitor positively and the left plate negatively. Thus, a counter voltage is established which acts in opposition to the applied voltage, and the current is the means by which the applied voltage and counter voltage tend to counterbalance each other.

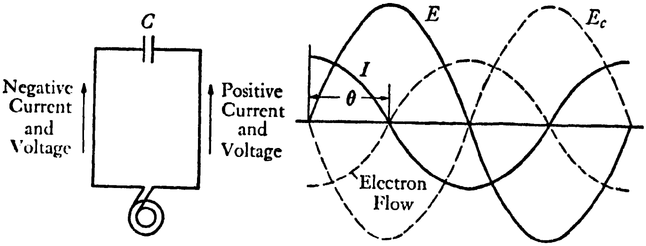


Fig. 1-13. Phase Relation Between Current and Voltage in an A-c Circuit Containing Only Capacitance.

As the applied voltage decreases from the maximum positive value to zero, the counter voltage of the capacitor causes the current to reverse.

During this period the applied voltage is positive but decreasing in value and the current is negative and increasing in value. At the instant the applied voltage wave passes through zero, the current reaches its maximum negative value and the capacitor is discharged.

During the period in which the applied voltage increases from zero to its maximum negative value, the direction of current remains negative, but its value decreases, and reaches zero at the instant the applied voltage passes through its negative maximum. The left plate of the capacitor is now charged positively and the right plate negatively.

As the applied voltage changes from its maximum negative value to zero, the current increases from zero to its positive maximum. Thus, *in an a-c circuit which contains only capacitance, the current leads the applied voltage by 90°.*

The wave of charging current in an a-c circuit containing only capacitance will be of cosine form if the impressed voltage wave is of sine form. This fact can be proved graphically or by use of the calculus. A cosine wave is similar to a sine wave in every respect except that its maximum value occurs at 0 degrees, and its zero value occurs at 90°.

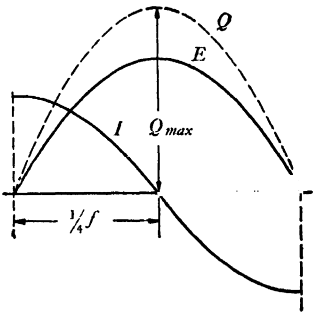


Fig. 1-14. Charge of a Capacitor on an A-c Source.

21. Charging Current in an A-c Circuit Containing Only Capacitance. The charge on a capacitor at any instant is given by the equation

$$q = Ce \qquad (1\text{-}19)$$

where q is the instantaneous value of quantity in coulombs (ampere-seconds), C is the capacitance in farads, and e is the instantaneous value of the applied voltage.

It is obvious then that when a sine-wave a-c voltage is impressed on a capacitor, the quantity wave and the voltage wave will be in phase. And:

$$Q_{max} = CE_{max}$$

From Fig. 1-14 it is also obvious that the maximum charge equals the product of the average current between zero and 90° and the time in seconds for the wave to pass from zero to 90°. That is:

$$Q_{max} = I_{ave} \times 1/4f$$

Then:

$$\begin{aligned} CE_{max} &= I_{ave} \times 1/4f \\ I_{ave} &= 4fCE_{max} \\ I_{ave} &= 0.636 I_{max} \\ 0.636 I_{max} &= 4fCE_{max} \\ I_{max} &= 4fCE_{max}/0.636 = 6.28fCE_{max} \end{aligned}$$

Multiplying both sides of the equation by 0.707 to obtain effective values of I and E, we have

$$(0.707 I_{max}) = 6.28 fC\,(0.707 E_{max})$$
$$I = 6.28 fCE = 2\pi fCE \tag{1-20}$$

Example 1-6: A 15-microfarad capacitor is placed across a 220-volt, 60-cycle a-c source. A charging current of 1.245 amp flows. Determine the maximum charge.

Solution. The time required to charge the capacitor is $1/4f$ or $1/240$ sec. The average value of the current from 90 degrees to zero degrees (assuming sine-wave form) is,

$$I_{ave} = I_{eff} \times 0.636/0.707 = 1.12 \text{ amp}$$
$$Q_{max} = I_{ave} \times 1/4f = 1.12 \times 1/240 = 0.00467 \text{ coulombs} \quad \textit{Ans.}$$

Alternate Solution.

$$Q_{max} = CE_{max}$$
$$E_{max} = E/0.707 = 311 \text{ volts}$$
$$Q_{max} = 0.000015 \times 311 = 0.00467 \text{ coulombs} \quad \textit{Ans.}$$

22. Capacitive Reactance. The effect of the counter voltage in opposing the change in current is called *capacitive reactance*, and is measured in ohms. Capacitive reactance equals the voltage applied to the capacitor divided by the charging current. Its symbol is X_C.

$$X_C = E/I \tag{1-21}$$

Substituting for I its value from equation 1-20 we have:

$$X_C = E/2\pi fCE = 1/2\pi fC = 1/\omega C \tag{1-22}$$

Since equation 1-20 is derived assuming ratios between average, effective, and maximum values as obtained from sine waves, it is evident that equation 1-22 is true only when the waves follow the sine law.

When C is expressed in microfarads, equation 1-22 may be changed to read:

$$X_C = 10^6/2\pi fC = 10^6/\omega C \tag{1-23}$$

Problem 1-32: The charge on a capacitor is 0.00964 coulomb after a charging period of 1/120 sec. What is the average rate of current flow into the capacitor?

Problem 1-33: A 40-microfarad capacitor is placed across a 240-volt, variable frequency a-c source. Determine the maximum charge when the frequency is 25 cycles; 60 cycles.

Problem 1-34: A 60-microfarad capacitor is placed across a 50-cycle a-c supply. A current of 1.8 amp flows. What voltage is applied to the terminals of the capacitor?

Problem 1-35: A capacitor is placed across a 220-volt, 60-cycle supply and a current of 1.56 amp flows in the circuit. What is the capacitance in microfarads?

Problem 1-36: A capacitor is placed across a 120-volt, 60-cycle source and a current of 0.94 amp flows in the circuit. Determine its reactance in ohms and its capacitance in mfd.

Problem 1-37: A 40-microfarad capacitor is placed across a 440-volt, 60-cycle supply. (*a*) Determine its capacitive reactance. (*b*) What current will flow? (*c*) If the frequency of the supply is changed to 40 cycles, what value of voltage is required to establish the same magnitude of charging current as obtained on 60 cycles?

23. Addition and Subtraction of Sine Waves. Fig. 1-15 shows an a-c series circuit containing resistance and inductance. The current I is in phase with the voltage across the resistor E_R, but lags the voltage across the inductive reactance E_L by 90°. Instantaneous values for plotting the applied voltage E are obtained by adding algebraically corresponding instantaneous values of E_R and E_L. Attention is called to the fact that θ, the phase angle between E and I, is an angle of lag between 0° and 90°. The actual value of θ depends on the ratio of resistance and inductive reactance in the circuit. Attention is also called to the fact that the maximum height of E in Fig. 1-15 is less than the arithmetical sum of the maximum heights of E_R and E_L.

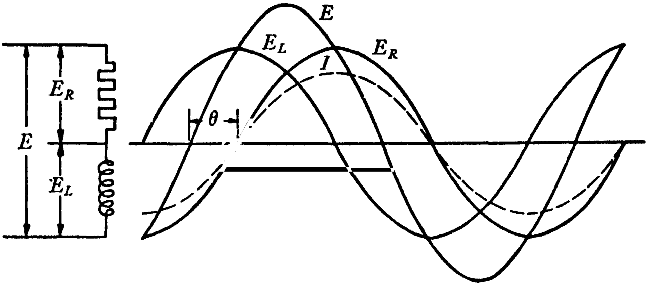


Fig. 1-15. Addition of Two Sine-Wave Voltages.

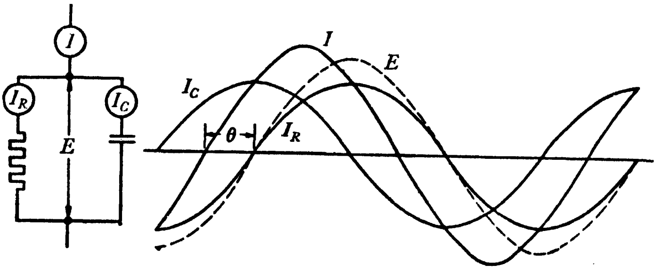


Fig. 1-16. Addition of Two Sine-Wave Currents.

In a similar manner it can be shown that a series circuit containing resistance and capacitance has a leading phase angle between 0° and 90°;

the actual value of the phase angle depending on the ratio of resistance to capacitive reactance.

One branch of the parallel circuit (Fig. 1-16) contains resistance and the other capacitance. The current I_R in the resistance branch is in phase with E, the voltage across the parallel circuit, and the current I_C in the branch containing capacitance leads E by 90°. Instantaneous values for plotting the wave of line current I are obtained by adding algebraically, instantaneous values of I_R and I_C. The phase angle between E and I is a leading angle of less than 90°, and the maximum height of I is less than the arithmetical sum of the maximum heights of I_R and I_C.

To subtract one quantity from another it is merely necessary to change its sign and add. Hence, to subtract one sine wave from another merely reverse the wave to be subtracted and add its instantaneous values to those of the other wave. The branch current I_1 (Fig. 1-17) is subtracted from the line current I to obtain the second branch current I_2.

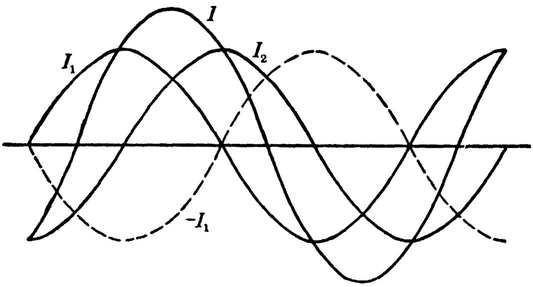


Fig. 1-17. Difference of Two Sine-Wave Currents.

Problem 1-38: One branch of a parallel circuit contains resistance and inductance. The current in this branch has a maximum value of 20 amp and lags the voltage across the parallel circuit by 30°. A second branch contains resistance and capacitance. The current in the second branch has a maximum value of 15 amp and leads the voltage by 30°. (*a*) Determine the maximum value of the line current. (*b*) Determine the phase angle between line current and voltage. Solve the problem by constructing sine waves and adding their instantaneous values. *Suggestion:* Place a piece of $8\frac{1}{2}$ by 11-inch cross-section paper with the long side horizontal. Construct semicircles with their centers on the left grid $3\frac{1}{2}$ inches from the bottom grid. Use scale 1 inch = 10 amp. The value of the voltage in this problem is of no particular value, but the position of the voltage wave is important. Assume some height for the voltage wave, say $2\frac{1}{2}$ inches maximum, and draw semicircle with radius using the same center as above. Lay out the horizontal time axis for 420° starting with an angle of − 30° and ending with an angle of + 30°. Use a scale for the time axis of 1/4 inch = 15°. Draw the voltage wave passing through its zero value at zero degrees. I_1 should be drawn 30^c behind E, and I_2, 30° ahead of E.

Problem 1-39: Two coils each containing resistance and inductance are connected in series across an a-c source. The voltage across coil one has a maximum value of 50, and the current lags the voltage across this coil by 30°. The voltage across coil two has a maximum value of 70 and the current lags the voltage across this coil by 60°. Construct sine curves and determine: (*a*) the maximum value of the voltage across the circuit; (*b*) the phase angle between the voltage across the circuit and the current. *Suggestion:* Place the long side of a piece of $8\frac{1}{2}$ by 11-inch cross-section paper horizontally, and construct three semicircles with their centers on the left grid $3\frac{1}{2}$ inches above the bottom grid. Use scale 1/4 inch = 10 volts for the voltage waves. The actual value of the current wave is not important, but its position is of value, hence, it may be constructed to any convenient scale, say 2 inches maximum height. Draw the horizontal time axis for 420 degrees, starting with 0° and ending with 60° for the second cycle, using a scale 1/4 inch = 15°. Draw the current wave passing through zero value at zero degrees. E_1 should pass through its corresponding zero value 30° ahead of I, and E_2 through its corresponding zero value 60° ahead of I.

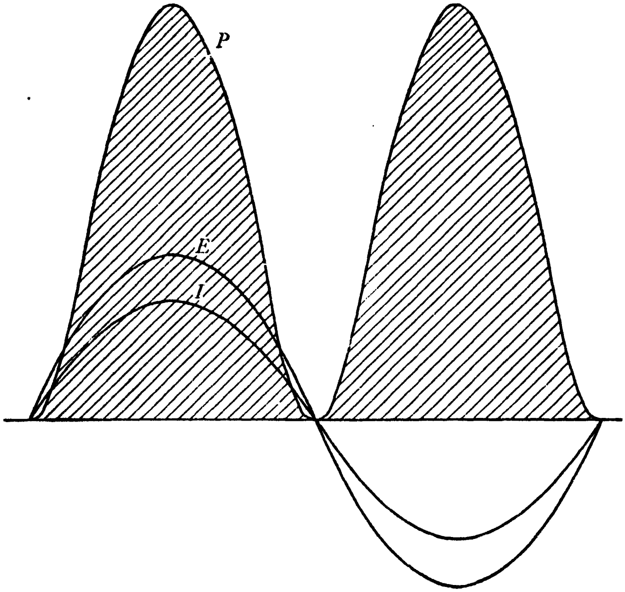


Fig. 1–18. Power Wave for an A-c Circuit Containing Only Resistance.

24. The Product of Two Sine Waves — Power and Power Factor. Power in a d-c circuit is the product of the voltage and current. Power in an a-c circuit is the product of effective voltage and effective current only when the current and voltage are in phase. Fig. 1–18 shows the power curve for an a-c circuit which contains only resistance. The points on the power curve are obtained by multiplying corresponding instantaneous values of voltage and current. If the area of the power curve for 360° is measured and divided by its length, the average power obtained will be found to equal the product of E and I.

The circuit (Fig. 1-19) contains only inductance, and the current lags the voltage by 90°. Instantaneous values for plotting the power curve are obtained by multiplying corresponding instantaneous values of E and I.

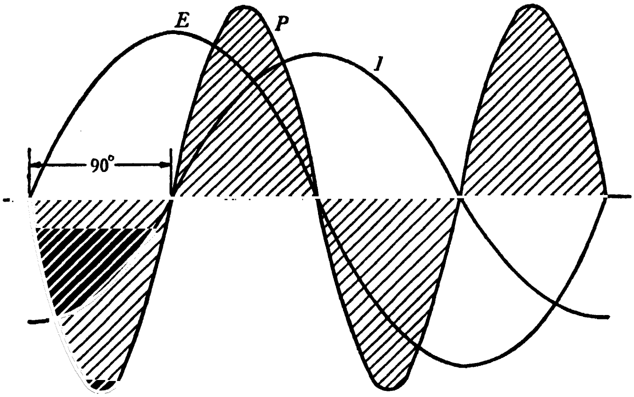


Fig. 1-19. Power Wave for an A-c Circuit Containing Only Inductance.

Attention is called to the equal positive and negative power loops, i.e., four loops each cycle, two in which power passes from the source to the circuit and two in which the circuit returns power to the source. The average power in a circuit of this type is zero.

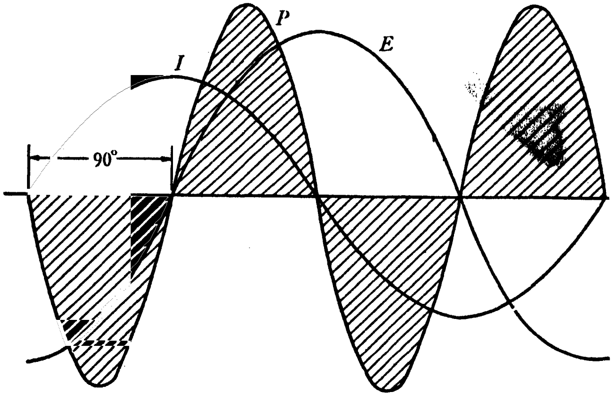


Fig. 1-20. Power Wave for an A-c Circuit Containing Only Capacitance.

Power in a circuit containing only capacitance is shown by Fig. 1-20. The phase angle is now a leading angle of 90°, and the average power is zero.

The circuit (Fig. 1-21) contains resistance and inductance in such proportions as to give a lagging phase angle of 60°. Instantaneous values for plotting the power curve are obtained by multiplying corresponding instantaneous values of E and I. When its area is measured and divided by its length, the average height of the power curve is found to be one-half the product of the effective values of E and I. That is, the product of E and I must be multiplied by some factor to obtain the power, and in this case the factor is 0.5.

The factor by which the volt-amperes of an a-c circuit must be multiplied to obtain the power is known as the *power factor* of the circuit. As the phase angle is increased from zero to 90° the algebraic sum of the plus and negative power loops grows smaller. At zero phase angle (Fig. 1-18) there is no negative loop and the average power is $E \times I$. In this case the power factor is unity. When the phase angle equals 90° (Figs. 1-19 and 1-20), the average power is zero, and the power factor is also zero. When the phase angle equals 60° (Fig. 1-21), the power factor is 0.5. In the cases illustrated, the power factor has equaled the *cosine of the phase angle*. This can be proved for any phase angle. Then the equation for power in an a-c circuit is:

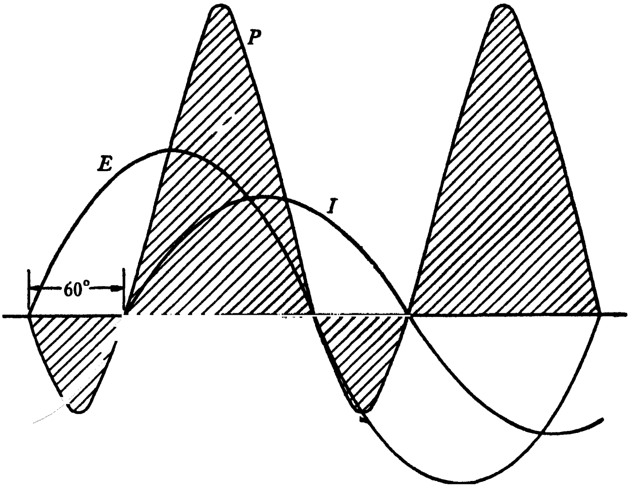


Fig. 1-21. Power Wave for an A-c Circuit with Current Lagging Voltage by 60°.

$$P = EI \cos \theta \tag{1-24}$$

where P is the average power in watts, E the effective value of voltage, I the effective value of current, and θ the phase angle.

The product of E and I in an a-c circuit is known as the volt-amperes of the circuit, and cos θ is the power factor. Transposing the terms of equation 1-24 we have:

$$\mathrm{Cos}\, \theta = P/EI \tag{1-25}$$

Problem 1-40: I_{max} equals 20 amp and lags E_{max} by 45°. E_{max} equals 25 volts. Construct sine waves of voltage and current and determine points on the power wave by multiplying corresponding instantaneous values of E and I. Measure the area under the power curve and determine its average height. Check the value obtained with that computed by use of equation 1-24. *Suggestion:* Use scales of 1 inch = 10 amp, 1 inch = 10 volts, 1 inch = 100 watts, and 1/4 inch = 15°. Draw curves with the long side of the $8\frac{1}{2}$ by 11-inch cross-section paper placed horizontally. Locate the centers of the semicircles for construction of voltage and current waves on the left grid, $2\frac{1}{2}$ inches above the bottom grid.

Problem 1-41: An ammeter, a voltmeter, and a wattmeter are inserted in the circuit of a single-phase induction motor and read 11 amperes, 230 volts, and 1840 watts. (*a*) Determine the power factor of the motor. (*b*) Determine the phase angle between the current and voltage.

Problem 1-42: A phase angle meter placed in an a-c circuit indicates an angle of 47°. The circuit takes 670 watts. What are the volt-amperes of the circuit?

25. Waves Other Than Sine Waves. The alternating voltage, current, and power waves discussed in this chapter have followed the sine law. Frequently, alternating voltage and current waves are distorted and contain harmonics which are waves of higher frequencies. In general, the methods that apply to sine waves will give sufficiently accurate results when applied to other wave forms. This is particularly true when the equivalent sine curve and equivalent phase angle are used. A-c ammeters and voltmeters read effective value regardless of wave form, and the wattmeter reads average power. The power factor is the wattage divided by effective volt-amperes, and the effective phase angle is taken as the angle whose cosine is the power factor. The value of this effective phase angle is the same as the actual phase angle when both current and voltage waves follow the sine law. The error in following this procedure is increased in capacitive circuits which tend to exaggerate harmonics. In inductive circuits which tend to suppress harmonics, the error is decreased. The discussions and problems which follow in this text are based upon the assumption that alternating-current quantities are of sine wave form.

QUESTIONS

1-1. State some of the reasons why alternating current is more generally used than direct current.

1-2. How is the instantaneous value of a sine wave at any angle obtained when the maximum value is known?

1-3. Define alternation, cycle, period, frequency.

1-4. Discuss electrical and space-degrees.

1-5. Explain how the average height of a sine wave may be obtained graphically.

1-6. How is the alternating-current ampere defined?

1-7. Explain how the rms value of a sine wave of current is obtained.

1-8. What is the ratio between the average and maximum value of a sine wave? Between the effective and maximum value?

1-9. When the a-c voltage applied to a motor is stated as 220 volts, what value of the voltage wave is to be understood?

1-10. Why does effective resistance differ from d-c resistance?

1-11. If the cross-section area of an iron core which is inserted in an inductively wound coil is increased, what effect does it have on the effective resistance of the coil? Why?

1-12. What is impedance in an a-c circuit?

1-13. How can a resistor be wound noninductively?

1-14. Distinguish between time angle and phase angle.

1-15. Explain why the current in an a-c circuit containing only inductance lags the voltage by 90°.

1-16. What is meant by the term "inductive reactance"? How does it differ from inductance; from the voltage of self-induction?

1-17. Explain why the current in an a-c circuit containing only capacitance leads the voltage by 90°.

1-18. What is meant by capacitive reactance? What is its symbol?

1-19. State the equation for determining power in an a-c circuit.

1-20. What is meant by power factor? What relationship does it have to the phase angle?

1-21. What is an angle of lag; an angle of lead?

1-22. Explain the difference between volt-amperes and watts.

PROBLEMS

1-43. The average value of a sine wave of voltage is 90 volts. What is its effective value?

1-44. How long in seconds is required for a 50-cycle sine wave to pass from zero to its average value?

1-45. How long in seconds is required for a 25-cycle wave to pass from its average value to its effective value?

1-46. The effective value of an alternating current is 200 amp. What is its instantaneous value at the 30° instant?

1-47. What is the period of the wave produced by an 8-pole alternator which is driven at 600 rpm?

1-48. A radio frequency wave requires 1/4,000,000 sec to pass from a zero to a maximum value. What is the frequency of the wave?

1-49. A rotating armature alternator has 24 poles. Through how many space-degrees must a conductor on its armature move in order for the wave form to pass from its average to its effective value?

1-50. How many space-degrees per cycle does a conductor on the surface of a rotating armature alternator move if the machine has 16 poles and is generating 25 cps?

1-51. A coil of negligible resistance has an inductance of 0.25 henry. Plot a curve showing the variation of inductive reactance for this coil with frequencies of 20, 30, 40, 50, and 60 cycles applied to the coil.

1-52. A coil is tested on a 120-volt, 60-cycle source, and draws 4.8 amp and 160 watts. What is the effective resistance of the coil?

1-53. A coil with an inductance of 0.35 henry is placed across a 25-cycle source. What is the effective value of counter-voltage induced in the coil when a current of 3-amp effective value is flowing?

1-54. A coil with an inductance of 0.62 henry and negligible resistance is placed across a 220-volt, 60-cycle source. (*a*) What is the reactance of the coil? (*b*) What current will flow in the coil? (*c*) What is the power required by the coil?

1-55. A coil of 600 turns is wound on a wooden ring with a cross-section diameter of 2 inches. The average length of the magnetic path through the ring is 30 inches. (*a*) What is the inductance of the coil? (*b*) What is the reactance of the coil on a 60-cycle supply?

1-56. A capacitor is connected across a 240-volt, 60-cycle source and draws 4.2 amp. (*a*) What is the capacitive reactance? (*b*) The capacitance? (*c*) What power is drawn by the capacitor? (*d*) What current would the capacitor draw on a 110-volt, 25-cycle source?

1-57. Determine the inductive reactance of a coil having 0.2 henry inductance on 30 cycles; 40 cycles; 50 cycles.

1-58. Determine the capacitive reactance of a 30-mfd capacitor on 30 cycles; 40 cycles; 50 cycles.

1-59. A single-phase alternator has a rating of 50 kva at a voltage of 240 volts. (*a*) What is the current rating of the machine? (*b*) What kw will it supply at unity power factor? (*c*) What kw will it supply at a leading power factor of 90 per cent? (*d*) What kw will it supply at a lagging power factor of 60 per cent?

1-60. A 50-kva transformer operating without load on the secondary takes 3.14 amp at 2400 volts. The power taken by the primary is 1500 watts. What is the power factor of the transformer under this condition?

1-61. A wattmeter has a maximum voltage rating of 300, a maximum current rating of 5 amp and a wattage range of 1200 watts. The meter is used on a 230-volt source. (*a*) Without exceeding the ratings of the coils, what is the maximum wattage which can be read at unity power factor? (*b*) At 60 per cent power factor? (*c*) At 30 per cent power factor? (*d*) If the meter were used to measure power supplied by a 220-volt source at a power factor of 50 per cent, what would be the percentage of overload on the current coil of the meter when it reads 1200 watts?

Laboratory Experiments

Experiment 1-1

Purpose. To determine the difference between ohmic and effective resistance in circuits with and without iron cores.

Procedure. 1. Connect the circuit as shown in Fig. 1-7. *Caution:* Place the instruments on the test table at a distance from the coil so that stray magnetic fields will not affect their indications. Place the iron core in the coil, insert all resistance, and apply 120 volts, 60-cycle, ac. Reduce the value of series resistance as much as possible without exceeding the current ratings of the instruments and the equipment. Read instruments and record data.

2. Insert all resistance and slowly remove the iron core, checking to see that the current does not rise too high as the core is removed. Reduce resistance until the current is the same as in part 1. Read instruments and record data. If necessary to obtain accurate readings, use a voltmeter and wattmeter of lower voltage range.

3. Disconnect the circuit from the a-c source, insert all resistance, and reconnect the circuit to the 120-volt d-c source. Replace the voltmeter and ammeter with d-c instruments. The wattmeter will not be required on direct current. Reduce resistance until the current is the same as in parts 1 and 2. Read instruments and record data.

Report. 1. Submit a complete report (see Appendix A).

2. Compute and record the value of effective resistance obtained in part 1, the resistance obtained with alternating current in part 2, and the ohmic resistance obtained with direct current in part 3.

3. Explain in the discussion of results, the reason for the difference between the effective resistance obtained in part 1 and the resistance with a-c voltage applied as obtained in part 2. Also, explain the reason why different values of voltage were required to establish the same value of current in parts 1, 2 and 3.

EXPERIMENT 1-2

Purpose. To determine the relationship between inductive reactance and frequency.

Procedure. 1. Connect the apparatus as illustrated in Fig. 1-22. *Caution:* Locate the instruments so they will not be affected by stray magnetic fields. The variable frequency may be obtained from an alternator which is driven by an adjustable speed d-c motor. The limits of frequency should be from about 35 to 70 cps. The frequency may be measured directly with a frequency meter, or may be computed from the rpm and number of poles of the alternator. Insert resistance and apply 120 volts at the lower frequency limit to the circuit. Reduce resistance until the current rises to a value which will give a reading high on the scale of the ammeter, but which will not exceed the current rating of the wattmeter or other equipment. Read and record data.

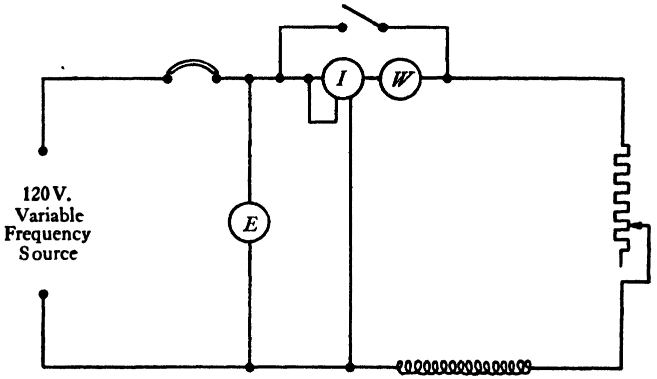


Fig. 1-22. Circuit Diagram for Experiment 1-2.

2. Without making further adjustments of resistance, repeat part 1 for eight to ten values of frequency between the low and high limits. The magnitude of the voltage should be the same for all values of frequency.

Report. 1. Submit a complete report (see Appendix A).

2. Compute and record on the data sheet for each value of frequency: $\cos\theta$, θ, Z, R_{eff}, X_L, and L. The value of X_L is computed from the following equation which will be explained later in this text.

$$X_L = \sqrt{Z^2 - R_{eff}^2}$$

3. Construct two graphs on the same sheet, using frequency as abscissa for both curves, inductive reactance as ordinate for one curve, and current as ordinate for the other curve.

4. In the discussion of results, state the relationship between the experimental findings and the theory given in Chapter I. From a theoretical point of view should the value of L in this experiment have been constant? Was it constant?

EXPERIMENT 1-3

Purpose. To determine the relationship between capacitive reactance and frequency.

Procedure. 1. Connect a capacitor across a 120-volt variable frequency source. Insert an ammeter in the circuit to read the charging current and a voltmeter to read the applied voltage. Energize the circuit with 120 volts at a frequency of about 70 cps, and check to see that the ammeter needle indicates high on the scale. If not, replace the ammeter with one of lower range. Read and record data.

2. Repeat part 1 for eight or ten values of frequency between the high limit and a low limit of about 35 cps. The magnitude of voltage should be constant for all readings.

Report. 1. Submit a complete report (see Appendix A).

2. Compute and record on the data sheet for each value of frequency: X_C and C.

3. Construct two graphs on the same sheet using frequency as abscissa for both curves, capacitive reactance as the ordinate for one curve, and current as the ordinate for the other curve. Use the same scales for abscissa and ordinates as used in experiment 1-2.

4. In the discussion of results state the relationship between the experimental results and the theory contained in Chapter I. Should the curve plotted between X_C and f be a straight line? Why?

CHAPTER II

VECTORS AND COMPLEX QUANTITIES

1. Vectors. In Chapter I, elementary a-c circuit problems were solved graphically by developing sine waves of the quantities involved, and by adding, subtracting, or multiplying instantaneous values of these waves, as the case required, to obtain the resultant wave. This method of dealing with sine-wave quantities is tedious and laborious, and is seldom used except to prepare the student for a clearer understanding of more practical methods. These more practical methods of solving a-c circuit problems have as their bases the representation of alternating current and voltage by *vectors*. A vector is a line drawn to scale with an arrow head placed on one end to indicate direction. Thus, a vector quantity has both magnitude and direction as contrasted with a scalar quantity which has only magnitude. Temperature, area, and volume are scalar quantities since they have only magnitude, but force, motion, voltage, current, and the like are vector quantities since they have direction as well as magnitude. The

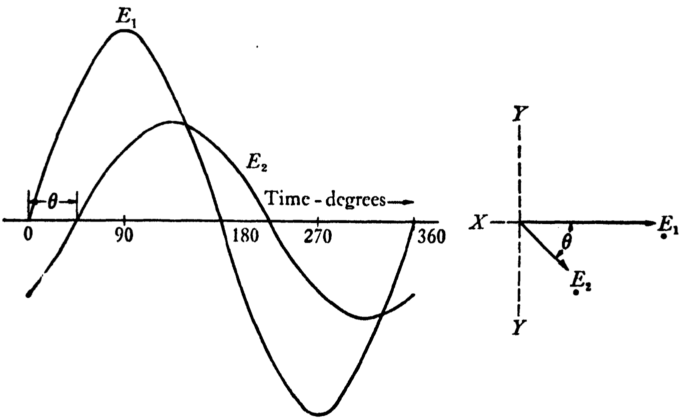


Fig. 2-1. Waves and Vectors of Two Alternating Voltages Displaced θ Time-Degrees from One Another.

magnitude of a vector quantity is represented by the length of the vector, and the direction in which the vector quantity acts is indicated by the direction of the arrow. Vector quantities are generally denoted by placing dots under their symbols, i.e., $\underset{\cdot}{I}$ is read as the vector I, and $\underset{\cdot}{E}$ as the vector E.

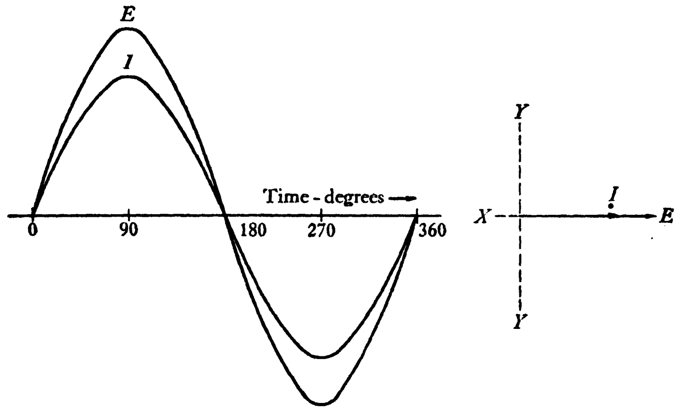
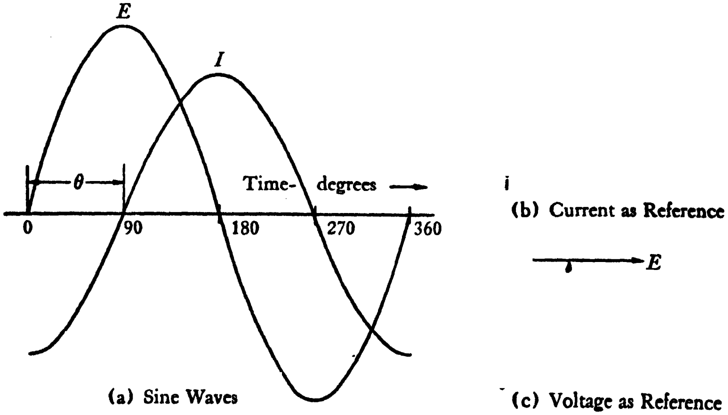


Fig. 2–2. Waves and Vectors of Alternating Current and Voltage in a Circuit Containing Only Resistance.

2. Vector Representation of Alternating Currents and Voltages. Alternating-current and voltage vectors are referred to a coordinate plane, which represents 360 electrical time-degrees. By agreement, *counterclockwise rotation of vectors is taken as positive and clockwise rotation as negative.* The horizontal axis extending to the right of the intersection of horizontal and vertical axes is taken as the *reference axis,* and current and voltage vectors are referred to this position.



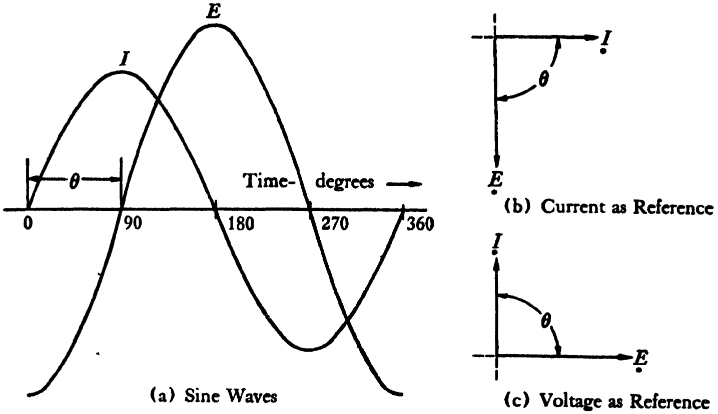
(a) Sine Waves

(b) Current as Reference

(c) Voltage as Reference

Fig. 2–3. Waves and Vectors of Alternating Current and Voltage in a Circuit Containing Only Inductance.

The vectors and sine waves of two voltages of the same frequency are shown in Fig. 2-1. E_2 has one-half the magnitude of E_1 and lags E_1 by θ degrees. If the instantaneous values of E_1 are plotted with the electrical time-degrees of the wave coinciding with the electrical time-degrees of the coordinate plane, the equation for the sine wave E_1 will be,



(a) Sine Waves

(b) Current as Reference

(c) Voltage as Reference

Fig. 2-4. Waves and Vectors of Alternating Current and Voltage in a Circuit Containing Only Capacitance.

$$e_1 = E_{1max} \sin \omega t$$

Now the sine wave E_2 passes through its zero value θ degrees after E_1 passes through the corresponding value, and if the same time axis is to be used for both waves, the equation of E_2 will have to be,

$$e_2 = E_{2max} \sin (\omega t - \theta)$$

For any instantaneous value of E_1, the corresponding instantaneous value of E_2 lies θ degrees behind, and the magnitude of any instantaneous value of E_2 is one-half the magnitude of the corresponding instantaneous value of E_1. Thus, $\dot{E}_1$ (Fig. 2-1) is a symbol of the sine wave E_1, and $\dot{E}_2$ is the symbol of a second sine wave which has the same frequency and one-half the magnitude of E_1, and which lags E_1 by θ degrees.

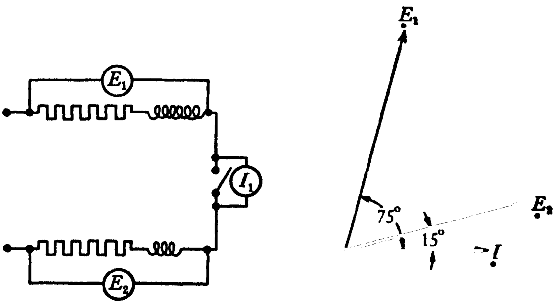


Fig. 2-5. Circuit and Vector Diagrams of Two Coils Connected in Series.

In Fig. 2-1, $\underset{\cdot}{E}_1$ and $\underset{\cdot}{E}_2$ represent the maximum values of the waves. They could just as well have represented any two corresponding instantaneous values. When a vector is used to represent a sine-wave quantity it is customary to allow its length to represent the effective value of the wave.

3. Vector Diagrams of A-c Series and Parallel Circuits. The current in a series circuit is common to all parts of the circuit. Hence, it is customary to plot the current in a series circuit along the reference axis. The various voltage drops are referred to the current. The common quantity for the parallel circuit is the voltage across the parallel branches. Thus, in a parallel circuit it is customary to plot the voltage along the horizontal positive axis, and the various branch currents are referred to the voltage.

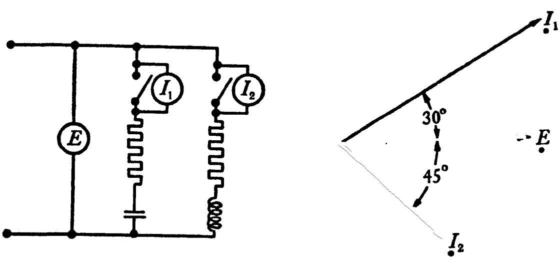


Fig. 2-6. Circuit and Vector Diagrams of a Parallel Circuit.

Fig. 2-2 shows the sine waves and corresponding vector diagram for an a-c circuit containing only resistance. Since current and voltage are in phase in this type of circuit, both $\underset{\cdot}{E}$ and $\underset{\cdot}{I}$ coincide with the reference axis. The lengths of the vectors represent to scale the effective values of current and voltage, although the scale used for current may differ from the scale for voltage.

Fig. 2-3 shows the sine waves and corresponding diagrams for an a-c circuit containing only inductance. In (*b*) the current vector is plotted along the reference axis, and in (*c*) the voltage is used as reference. The current lags the voltage by 90° in this type of circuit, hence the current vector is drawn 90° in a clockwise or negative direction from the voltage vector. Fig. 2-4 shows the vector diagrams for an a-c circuit containing only capacitance. The current now leads the voltage by 90°, and hence the current vector is drawn 90° in a counterclockwise or positive direction from the voltage vector.

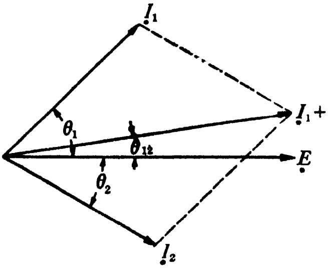


Fig. 2-7. Vector Sum, Parallelogram Method.

Fig. 2-5 gives the circuit and vector diagrams for a series circuit containing two coils, each having inductance and resistance. The current $\underset{\cdot}{I}$ lags the voltage across the first coil $\underset{\cdot}{E}_1$ by 75°, and lags the voltage across the second coil $\underset{\cdot}{E}_2$ by 15°. Since the same current flows in both coils, the circuit being in series, the current vector is drawn at the zero angle for reference. $\underset{\cdot}{E}_1$ is then plotted 75° ahead of the current, and $\underset{\cdot}{E}_2$ 15° ahead.

Branch one of the parallel circuit (Fig. 2-6) contains resistance and capacitance in such ratio as to cause $\underset{\cdot}{I}_1$ to lead $\underset{\cdot}{E}$ by 30°. Branch two contains resistance and inductance, and $\underset{\cdot}{I}_2$ lags $\underset{\cdot}{E}$ by 45°. $\underset{\cdot}{E}$ is plotted along the reference axis.

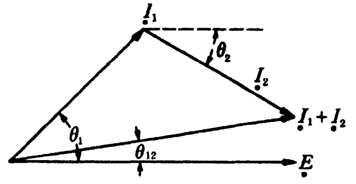


Fig. 2-8. Vector Sum, Triangle Method.

4. Graphical Methods of Determining Vector Sum and Difference. The sum of two vectors may be obtained graphically by the parallelogram method as shown in Fig. 2-7. A broken line parallel and equal in length to $\underset{\cdot}{I}_1$ is drawn from the tip of $\underset{\cdot}{I}_2$, and a second broken line parallel and equal in length to $\underset{\cdot}{I}_2$ is drawn from the tip of $\underset{\cdot}{I}_1$. These broken lines together with the vectors $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$ form a parallelogram. The diagonal drawn from the origin to the point where the two broken lines intersect is the vector sum of $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$. The vector sum of $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$ may also be obtained graphically by a triangle diagram as illustrated in Fig. 2-8. In this case the origin of vector $\underset{\cdot}{I}_2$ is located at the end of $\underset{\cdot}{I}_1$. The vector from the origin of $\underset{\cdot}{I}_1$ to the end of $\underset{\cdot}{I}_2$ is the vector sum of $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$. The voltage vector is placed in the diagrams as a reference about which the current vectors are plotted. The angles shown are the phase angles between the various currents and the voltage.

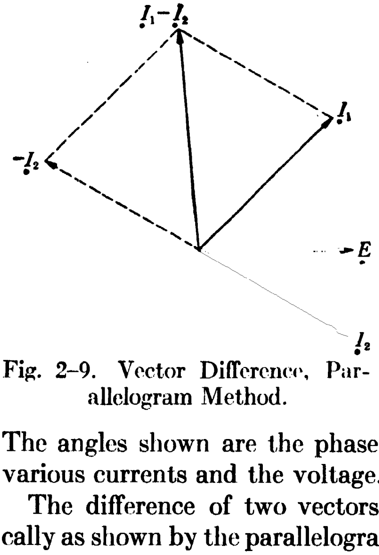


Fig. 2-9. Vector Difference, Parallelogram Method.

The difference of two vectors is obtained graphically as shown by the parallelogram method in Fig. 2-9. The vector $\underset{\cdot}{I}_2$ is reversed to give $-\underset{\cdot}{I}_2$ and this value is added to $\underset{\cdot}{I}_1$ to give the vector difference of $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$. To obtain vector difference by the triangle method (Fig. 2-10), $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$ are plotted with their origins at the same point. A vector drawn from the end of $\underset{\cdot}{I}_2$ to the end of $\underset{\cdot}{I}_1$ will be the difference of $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$.

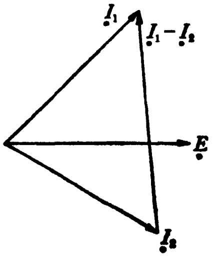


Fig. 2-10. Vector Difference, Triangle Method.

By using the parallelogram diagram for vector sum and the triangle diagram for vector difference (Fig. 2-11), it is apparent that the diagonal of the parallelogram which intersects the angle between two vectors is their vector sum, and the diagonal of the parallelogram opposite the angle between the two vectors is their vector difference.

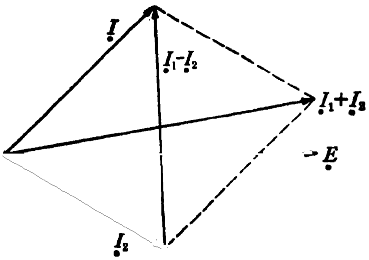
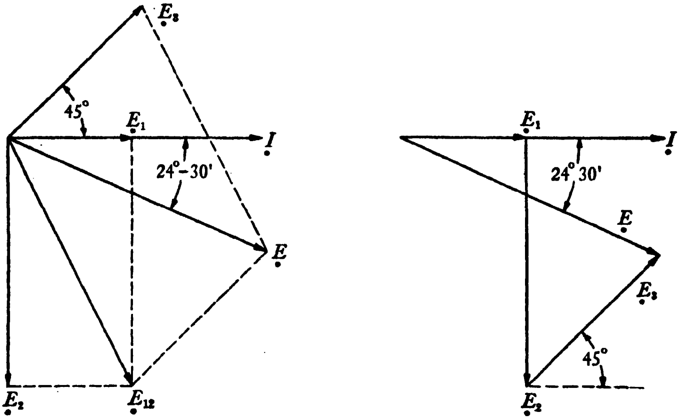


Fig. 2-11. Vector Sum and Difference.

Example 2-1: An a-c series circuit contains a noninductive resistor, a capacitor, and a coil having both resistance and inductance. The voltage E_1 across the resistor is 40 volts, E_2 across the capacitor 80 volts, and E_3 across the coil 60 volts. The current lags the voltage across the coil by 45°.. Determine the voltage across the circuit, and the angle between the line current and line voltage.

Solution. The current vector is plotted to a convenient length along the reference axis. E_1, E_2 and E_3 are plotted to a scale of 1 inch = 60 volts. E_1 and E_2 are added to give E_{12} and this in turn is added to E_3 to give the total voltage E. Fig. 2-12 shows both the parallelogram and the polygon diagrams for this problem. The value of E to scale is 90.7 volts, and θ is 24° 30′. *Ans.*



(*a*) Parallelogram Method. (*b*) Polygon Method.

Fig. 2-12. Diagrams for Example 2-1.

Example 2-2: An a-c parallel circuit having three branches contains resistance in branch one, inductance in branch two, and capacitance in branch three. I_1 equals 30 amp, I_2 equals 25 amp, and I_3 equals 15 amp. Determine the line current and the angle between line current and line voltage.

Solution. The vector diagram (Fig. 2-13) illustrates the solution of this problem. Since this is a parallel circuit with several currents, the voltage E

across the parallel circuit is plotted along the reference axis. $\underset{\bullet}{I}_1$, $\underset{\bullet}{I}_2$ and $\underset{\bullet}{I}_3$ are plotted to a scale of 1 inch = 20 amp. $\underset{\bullet}{I}_2$ and $\underset{\bullet}{I}_3$ are added to give $\underset{\bullet}{I}_{23}$, and this value is then added to $\underset{\bullet}{I}_1$ to give the total current $\underset{\bullet}{I}$. The value of $\underset{\bullet}{I}$ to scale is 31.6 amp, and θ is 18° 26′. *Ans.*

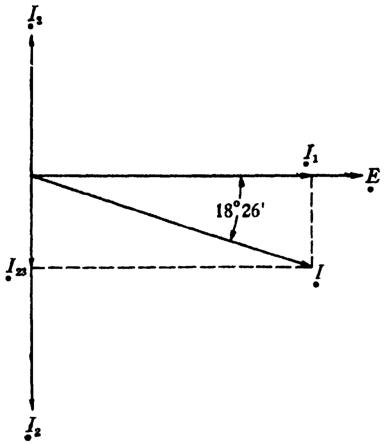

$\underset{\bullet}{I}_3$
$\underset{\bullet}{I}_1$
$\underset{\bullet}{E}$
18°26'
$\underset{\bullet}{I}_{23}$
$\underset{\bullet}{I}$
$\underset{\bullet}{I}_2$

Fig. 2–13. Vector Diagram for Example 2–2.

Example 2-3: A coil having 0.5 lagging power factor is connected in series with a resistor across a 110-volt a-c source. The voltage across the coil is 76 volts and the power factor of the entire circuit is 0.8 lag. What is the voltage across the resistor?

Solution. Fig. 2-14 shows the solution to this problem. Since this is a series circuit, the current vector is plotted along the reference axis. The phase angle of the coil is an angle whose cosine is 0.5, i.e., $\theta_1 = 60°$, and the phase angle of the entire circuit is an angle whose cosine is 0.8, i.e., $\theta = 36° \, 52'$. The voltage across the circuit $\underset{\bullet}{E}$, and the voltage across the coil $\underset{\bullet}{E}_1$ are plotted to a scale of 1 inch = 60 volts. Then $\underset{\bullet}{E}_2$, the voltage across the resistor, is the vector difference of $\underset{\bullet}{E}$ and $\underset{\bullet}{E}_1$. $\underset{\bullet}{E}_2$ equals 50 volts to scale. *Ans.*

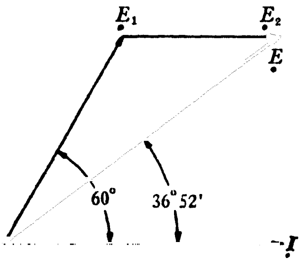


Fig. 2–14. Vector Diagram for Example 2–3.

Problem 2-1: A series circuit contains a noninductive resistor, a reactor of negligible resistance, and a capacitor. When connected across an a-c source the voltage across the resistor is 80 volts, across the reactor 100 volts, and across the capacitor 160 volts. Draw the vector diagram of this circuit, using a scale of 1 inch = 80 volts. The current vector should be drawn along the reference axis and may be any convenient length.

Problem 2-2: Determine graphically the value of the voltage across the series circuit, problem 2-1. Also, determine the phase angle between line voltage and current.

Problem 2-3: Three a-c circuits are connected in parallel. Circuit 1 draws 12 amp and has a lagging power factor of 0.8. Circuit 2 draws 8 amp at 0.6 leading power factor. Circuit 3 draws 10 amp at 0.3 lagging power factor. Draw the vector diagram of the circuit using a scale of 1 inch = 8 amp. The vector of voltage across the parallel circuit should be along the reference axis and may be any convenient length.

Problem 2-4: Determine graphically the line current in problem 2-3. Also, determine the phase angle between line current and the voltage across the parallel circuit.

Problem 2-5: Two voltages are 60° apart and each is equal to 50 volts. (*a*) What is their vector sum? (*b*) What is their vector difference?

Problem 2-6: Two voltages E_1 and E_2 are 120° apart and each equals 100 volts. (*a*) What is the vector sum of the two voltages? (*b*) What is their vector difference?

Problem 2-7: A coil having both inductance and resistance is placed in series with a resistance across a 220-volt a-c source. The power factor of the coil is 75 per cent lagging, and it is desired to limit the voltage across the coil to 150 volts. (*a*) Determine the voltage across the resistor. (*b*) Determine the power factor of the circuit. *Suggestion:* Plot the coil voltage to a selected scale at its correct angle. From the end of the coil voltage draw a horizontal line to the right. With the origin of the diagram as a center, and a radius of 220 volts to scale, strike an arc which intersects the horizontal line. A vector from the origin to this intersection will represent the line voltage. A vector plotted horizontally from the origin to the right, and equal to the length of the horizontal line from the end of the coil voltage vector to the intersection, represents the voltage across the resistor.

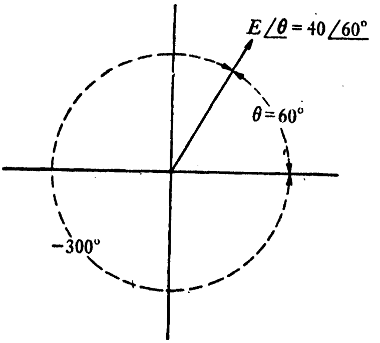


Fig. 2-15. Polar Vector.

Problem 2-8: A coil having a lagging power factor of 90 per cent, and drawing 7.5 amp, is connected in parallel with a capacitor across an a-c source. Determine graphically the current which must flow in the branch containing the capacitor to cause the line current to be in phase with the voltage across the parallel circuit.

5. Vectors in Polar Coordinates. Vectors which lie in the coordinate plane may be defined by their magnitude and the angle which they make with the reference axis (horizontal positive). Vectors defined in this manner are referred to as *polar vectors*. The magnitude of a polar vector is known as the *modulus*, and the direction angle as the *argument*. The modulus and argument of a polar vector are not a product, and, therefore, should not be treated as a product.

Multiplication, division, and many other mathematical operations may be carried out in a simple manner by use of polar vectors. However, *to add or subtract, polar vectors must first be changed to rectangular vectors.*

In Fig. 2-15 the polar vector $E\underline{/\theta}$ is given as $40\underline{/60^\circ}$, i.e., the vector has a modulus of 40 volts and lies in the first quadrant, making a positive angle of 60° with the reference axis. $E\underline{/\theta}$ also equals $E\overline{\backslash 360 - \theta}$, i.e., $40\underline{/60^\circ}$ is the same as $40\overline{\backslash 300^\circ}$. The symbol $\underline{/\ \ }$ indicates positive displacement, and $\overline{\backslash\ \ }$ negative displacement.

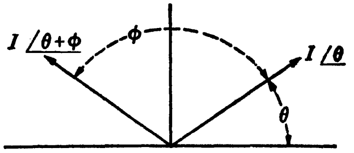


Fig. 2-16. Rotation of a Polar Vector.

To rotate a polar vector through an angle simply requires that the angle of displacement be added to the argument of the vector. The modulus will remain the same. That is, $I\underline{/\theta}$ rotated through ϕ degrees equals $I\underline{/\theta + \phi}$ (Fig. 2-16). A polar vector may be reversed by rotating the vector through ± 180°. Let it be required to reverse the polar vector $50\underline{/135^\circ}$. Then, as shown in Fig. 2-17,

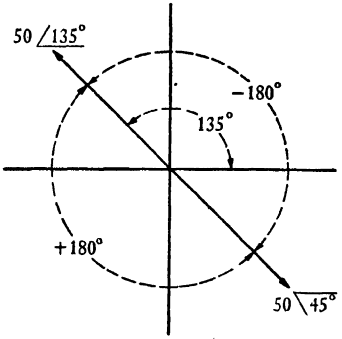

$50\underline{/135^\circ}$
-180°
135°
$+180^\circ$
$50\overline{\backslash 45^\circ}$

Fig. 2-17. Reversing a Polar Vector.

$$-(50\underline{/135^\circ}) = 50\underline{/135^\circ + 180^\circ} = 50\underline{/315^\circ}$$

or

$$-(50\underline{/135^\circ}) = 50\underline{/135^\circ - 180^\circ} = 50\overline{\backslash 45^\circ}$$

Problem 2-9: Rotate the following polar vectors: (*a*) $22\underline{/74^\circ}$ through $\underline{/150^\circ}$; (*b*) $128\underline{/30^\circ\ 40'}$ through $\overline{\backslash 62^\circ\ 38'}$; (*c*) $14\overline{\backslash 44^\circ\ 18'}$ through $\underline{/120^\circ}$; (*d*) $3.24\overline{\backslash 98^\circ\ 3'}$ through $\overline{\backslash 18^\circ\ 24'}$.

Problem 2-10: Reverse the following polar vectors:

(*a*) $0.065\overline{\backslash 46^\circ}$; (*b*) $2.84\underline{/38^\circ\ 16'}$; (*c*) $12.4\overline{\backslash 134^\circ\ 32'}$; (*d*) $68.4\underline{/148^\circ\ 26'}$

6. Vectors in Rectangular Coordinates.

A vector which is defined in terms of its horizontal and vertical components (rectangular coordinates) usually is referred to as a *rectangular vector.* That is, the vector is stated as a binomial or *complex quantity,* with the horizontal component as the first term and the vertical component as the second. The quadrant in which the vector lies is defined by the signs which precede the components. A rectangular vector having $+H$ and $+V$ components would lie in the first quadrant; $-H$ and $+V$ in the second; $-H$ and $-V$ in the third; and $+H$ and $-V$ in the fourth. The vertical component is distinguished from the horizontal component by use of the symbol j which is called a *complex operator*. The vector $\underset{\cdot}{I}$ (Fig. 2-18) expressed as a rectangular vector would be,

$$\underset{\cdot}{I} = I_H + jI_V \tag{2-1}$$

7. The Operator of Complex Algebra. The vector $\underset{\cdot}{I}$ (Fig. 2-19) lies along the reference axis. If we multiply $\underset{\cdot}{I}$ by -1, we have $-I$ which lies along the horizontal axis to the left. By operating on $\underset{\cdot}{I}$ with the value -1 we have caused the vector to rotate through 180°. The value -1 has two equal factors: $\sqrt{-1}\sqrt{-1}$; therefore, by operating on $\underset{\cdot}{I}$ with the two factors $\sqrt{-1}\ \sqrt{-1}$ we would likewise cause the vector to rotate through a positive angle of 180°. Now if $\underset{\cdot}{I}$ is operated on by one of the equal factors, $\sqrt{-1}$, the vector would be rotated through 90° in a counterclockwise direction; i.e., $\sqrt{-1}\ I$ would lie along the positive vertical axis. The value -1 also equals $(-\sqrt{-1})(-\sqrt{-1})$, and by operating on $\underset{\cdot}{I}$ with these factors the vector would be rotated through a negative angle of 180°. Operating on $\underset{\cdot}{I}$ with one of these factors, $-\sqrt{-1}$, would cause the vector to rotate 90° in a clockwise direction. That is, $-\sqrt{-1}\ I$ would lie along the negative vertical axis.

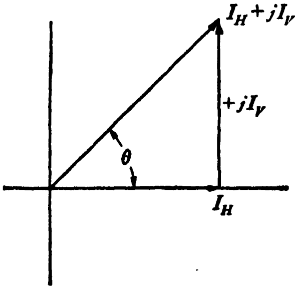


Fig. 2-18. Vector in Rectangular Coordinates.

The value $\sqrt{-1}$, which causes a vector to rotate counterclockwise through 90°, is known as an operator. In mathematics this operator is usually denoted by the symbol i, but in electrical engineering the symbol j is used to prevent confusion with the symbol of current. Likewise, $-\sqrt{-1}$ is represented by $-j$, and causes a vector to rotate 90° in a clockwise direction. Fig. 2-20 shows a rectangular vector in each quadrant.

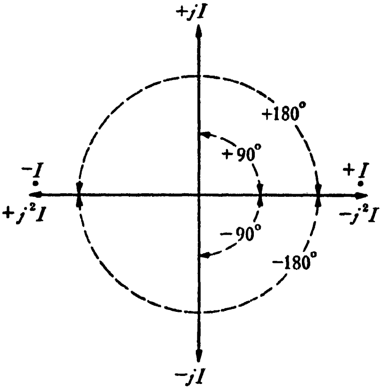


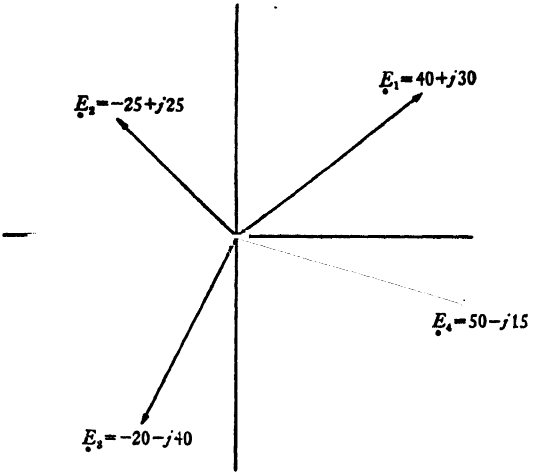
Fig. 2-19. Positions of Current Vector When Operated on by $\pm j$ and $\pm j^2$.

The operators $\sqrt{-1}\sqrt{-1}$ and $(-\sqrt{-1})(-\sqrt{-1})$ cause rotation through 180° and are represented by the operator j^2; that is, j^2 equals -1. Then

$$j^2\underset{\cdot}{I} = -\underset{\cdot}{I} \tag{2-2}$$

and

$$j^2(-\underset{\cdot}{I}) = -j^2\underset{\cdot}{I} = \underset{\cdot}{I} \tag{2-3}$$



$\underset{\cdot}{E}_1 = 40 + j30$
$\underset{\cdot}{E}_2 = -25 + j25$
$\underset{\cdot}{E}_3 = -20 - j40$
$\underset{\cdot}{E}_4 = 50 - j15$

Fig. 2-20. Vectors in the Four Quadrants Expressed in Rectangular Coordinates.

A rectangular vector may be reversed by multiplying by -1. In Fig. 2-21

$$\underset{\cdot}{I} = I_H + j\,I_V$$

Then

$$-\underset{\cdot}{I} = -I_H - j\,I_V$$

The plane in which rectangular vectors lie is known as the *complex plane.* The horizontal axis of the complex plane is called the *axis of reals*, and the vertical axis the *axis of imaginaries.*

Example 2-4: (*a*) Reverse the rectangular vector $8 - j\,6$; (*b*) Express $48 + j\,30 - j^2\,16$ as a rectangular vector.

Solution.

(*a*)
$$\underset{\cdot}{I} = 8 - j\,6$$
$$-\underset{\cdot}{I} = -8 + j\,6 \qquad \textit{Ans.}$$

(*b*) Substituting for $-j^2\,16$ its value $+16$ and simplifying, we have

$$48 + j\,30 + 16 = 64 + j\,30 \qquad \textit{Ans.}$$

Problem 2-11: Reverse the following: (*a*) $-24 + j\,11$; (*b*) $18 - j\,32$; (*c*) $-86.6 - j\,50$.

Problem 2-12: Express the following as rectangular vectors: (*a*) $15 + j\,54 + j^2\,48$; (*b*) $-54 + j\,18 + j^2\,12$; (*c*) $33 - j\,9 + j^2\,17$.

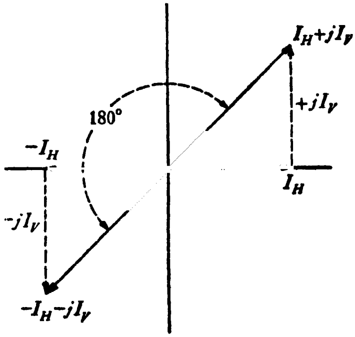


Fig. 2-21. Reversing a Vector Expressed in Rectangular Coordinates.

8. Relation Between Vectors Expressed in Rectangular and Polar Coordinates. A vector in polar form may be changed to rectangular form as shown in Fig. 2-22. The equation for changing from polar to rectangular form is

$$I\underline{/\theta} = I\cos\theta + jI\sin\theta \tag{2-4}$$

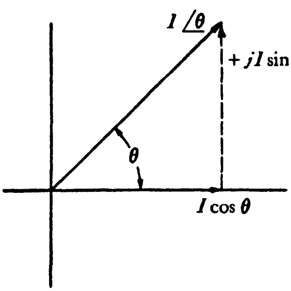


Fig. 2-22. Vector Changed from Polar to Rectangular Form.

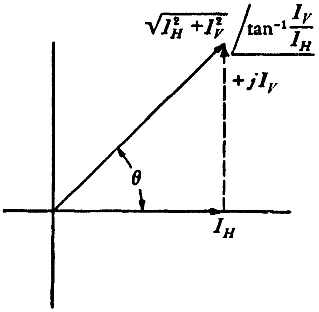


Fig. 2-23. Vector Changed from Rectangular to Polar Form.

The rectangular vector $I_H + jI_V$ (Fig. 2-23) may be expressed in polar form as follows:

$$I_H + jI_V = \sqrt{I^2_H + I^2_V}\underline{/\tan^{-1} I_V/I_H} \tag{2-5}$$

Example 2-5: Express the following polar vectors as rectangular vectors:

(*a*) $220\underline{/120^\circ}$; (*b*) $3.2\overline{\backslash 33^\circ\,18'}$; (*c*) $440\overline{\backslash 150^\circ}$

Solution.

(*a*) $220\underline{/120^\circ} = 220\,(\cos 120^\circ) + j\,220\,(\sin 120^\circ)$
$= -\,110 + j\,190.52$ *Ans.*

(*b*) $3.2\overline{\backslash 33^\circ\,18'} = 3.2\cos(-\,33^\circ\,18') + j\,3.2\sin(-\,33^\circ\,18')$
$= 2.675 - j\,1.757$ *Ans.*

(*c*) $440\overline{\backslash 150^\circ} = 440\cos(-\,150^\circ) + j\,440\sin(-\,150^\circ)$
$= -\,379.94 - j\,220$ *Ans.*

Example 2-6: Change the following rectangular vectors to polar vectors:

(*a*) $-\,25 + j\,25$; (*b*) $-20 - j\,40$; (*c*) $50 - j\,15$.

Solution.

(*a*) $-\,25 + j\,25 = \sqrt{(-\,25)^2 + 25^2}\underline{/\tan^{-1} 25/-\,25}$
$= 35.35\underline{/135^\circ}$ *Ans.*

(*b*) $-\,20 - j\,40 = \sqrt{(-\,20)^2 + (-\,40)^2}\underline{/\tan^{-1} -\,40/-\,20}$
$= 44.7\underline{/243^\circ\,26'}$ *Ans.*
$= 44.7\overline{\backslash 116^\circ\,34'}$ *Ans.*

(*c*) $50 - j\,15 = \sqrt{50^2 + (-\,15)^2}\underline{/\tan^{-1} -\,15/50}$
$= 52.1\underline{/343^\circ\,18'}$ *Ans.*
$= 52.1\overline{\backslash 16^\circ\,42'}$ *Ans.*

Problem 2-13: Express the following as rectangular vectors: (*a*) $18\backslash\overline{34^\circ\,15'}$; (*b*) $4.65\underline{/24^\circ\,12'}$; (*c*) $2300\backslash\overline{120^\circ}$.

Problem 2-14: Express the following as polar vectors: (*a*) $-62.5 - j\,16.8$; (*b*) $0.0024 - j\,0.0016$; (*c*) $-172 + j\,84$.

9. Addition and Subtraction of Vectors.

Polar vectors must first be changed to rectangular vectors before they can be added or subtracted. Rectangular vectors are added or subtracted according to the rules of algebra.

Example 2-7: Add the three voltages $60 + j\,20$; $-50 + j\,30$; and $-40 - j\,70$.

Solution. See Fig. 2-24.

$$\begin{array}{r} 60 + j\,20 \\ -\,50 + j\,30 \\ \underline{-\,40 - j\,70} \\ -\,30 - j\,20 \end{array} \qquad \textit{Ans.}$$

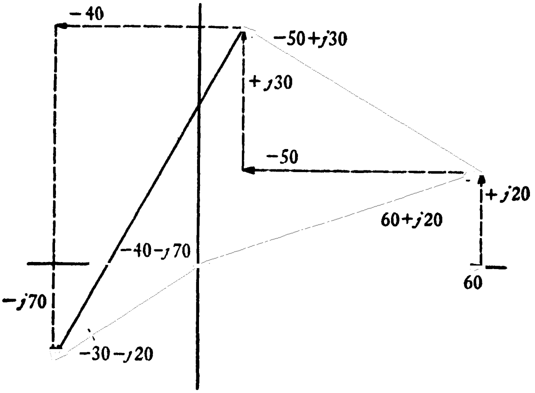


Fig. 2–24. Vector Diagram for Example 2–7.

Expressed in polar coordinates,

$$\begin{aligned} E\underline{/\theta} &= \sqrt{(-30)^2 + (-20)^2}\,\underline{/\tan^{-1} -20/-30} \\ &= 36.1\backslash\overline{146^\circ\,18'} \qquad \textit{Ans.} \end{aligned}$$

Example 2-8: Subtract $\underset{\cdot}{I}_1 = 12\underline{/45^\circ}$ amp from $\underset{\cdot}{I}_2 = 20\backslash\overline{30^\circ}$ amp.

Solution. See Fig. 2-25.

$$\begin{aligned} 12\underline{/45^\circ} &= 12\,(\cos 45^\circ + j\sin 45^\circ) = 8.48 + j\,8.48 \\ 20\backslash\overline{30^\circ} &= 20\,(\cos 30^\circ - j\sin 30^\circ) = 17.32 - j\,10 \end{aligned}$$

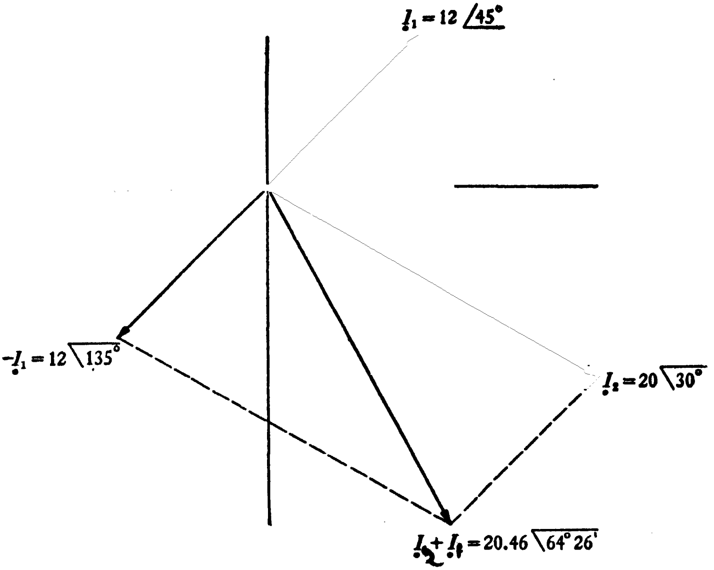

$\dot{I}_1 = 12\underline{/45^\circ}$
$-\dot{I}_1 = 12\overline{\backslash 135^\circ}$
$\dot{I}_2 = 20\overline{\backslash 30^\circ}$
$\dot{I}_1 + \dot{I}_2 = 20.46\overline{\backslash 64^\circ 26'}$

Fig. 2-25. Vector Diagram for Example 2-8.

To subtract, change the sign and add:

$$\begin{array}{r} 17.32 - j\,10 \\ \underline{-\ 8.48 - j\ \ 8.48} \\ 8.84 - j\,18.48 \end{array}$$

Expressed in polar coordinates,

$$\begin{aligned} \underline{I/\theta} &= \sqrt{8.84^2 + (-\,18.48)^2}\,\underline{/\tan^{-1} -\, 18.48/8.84} \\ &= 20.46\overline{\backslash 64^\circ\, 26'} \end{aligned}$$ *Ans.*

Example 2-9: A coil is placed in series with a single-phase motor to reduce the voltage across the motor terminals. The voltage across both motor and coil is $130\underline{/45^\circ}$ while that across the motor alone is $90\underline{/30^\circ}$. Determine the voltage across the coil.

Solution. See Fig. 2-26. The voltage across the coil equals the voltage across the line minus the voltage across the motor. The polar vectors must be changed to rectangular vectors.

$$\begin{aligned} 130\underline{/45^\circ} &= 130\,(\cos 45^\circ + j \sin 45^\circ) \\ &= 91.9 + j\,91.9 \text{ volts} \\ 90\underline{/30^\circ} &= 90\,(\cos 30^\circ + j \sin 30^\circ) \\ &= 78 + j\,45 \text{ volts} \end{aligned}$$

To subtract, change signs and add:

$$\begin{array}{r} 91.9 + j\,91.9 \\ \underline{-\ 78.0 - j\,45.0} \\ 13.9 + j\,46.9 \end{array}$$

$$\text{coil voltage} = 48.9\underline{/73^\circ\,29'}\ \text{volts} \qquad \textit{Ans.}$$

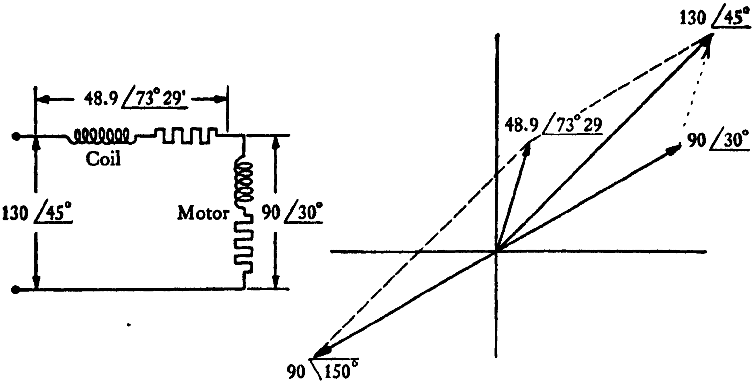


Fig. 2–26. Diagrams for Example 2–9.

Problem 2-15: Add $-75 + j\,25$ and $35 - j\,48$. Express the answer in polar form.

Problem 2-16: Subtract $64\overline{\backslash 18^\circ}$ from $136\underline{/128^\circ}$ and express answer in polar form

Problem 2-17: Add $28\underline{/15^\circ}$ to $43\overline{\backslash 144^\circ}$ and express answer in polar form.

Problem 2-18: Subtract $-44 + j\,16$ from $216 - j\,64$ and express answer in polar form.

Problem 2-19: An a-c parallel circuit contains three branches. The current in branch one is $8.5\overline{\backslash 28^\circ}$ amp; in branch two $4.8\underline{/16^\circ}$ amp; and in branch three $5.3\overline{\backslash 64^\circ}$ amp. Determine the line current and express it in polar coordinates.

Problem 2-20: A resistor, a coil having both resistance and inductance, and a capacitor are connected in series across an a-c source. The voltage across the resistor is $36\underline{/0^\circ}$ volts; across the coil $43\underline{/58^\circ}$ volts; and across the capacitor $94\overline{\backslash 90^\circ}$ volts. Determine the voltage applied to the series circuit and express the answer in polar coordinates.

Problem 2-21: A parallel circuit has two branches. The line current is $24.2\overline{\backslash 17^\circ\,14'}$ amp; and the current in branch one is $14.7\overline{\backslash 54^\circ\,26'}$ amp. Determine the current in branch two and express the answer in polar coordinates.

10. Multiplication of Vectors. The product of two polar vectors equals the product of their moduli and the sum of their arguments.

Example 2-10: Multiply $8\underline{/30^\circ}$ by $6\overline{\backslash 40^\circ}$

Solution. $8\underline{/30^\circ} \cdot 6\overline{\backslash 40^\circ} = (8 \times 6)\underline{/30^\circ + (-40^\circ)}$

$$= 48\overline{\backslash 10^\circ} \qquad \textit{Ans.}$$

The product of two rectangular vectors is obtained according to the rules of algebra.

Example 2-11: Multiply $12 + j\,6$ and $8 - j\,3$.

Solution.

$$\begin{array}{l} 12 + j\ \ 6 \\ \underline{\ 8 - j\ \ 3} \\ 96 + j\,48 \\ \underline{\qquad - j\,36 - j^2\,18} \\ 96 + j\,12 - j^2\,18 \end{array}$$

Substituting for $- j^2\,18$ its value $+ 18$, we have,

$$114 + j\,12 \qquad \textit{Ans.}$$

Problem 2-22: Multiply (*a*) $7\underline{/150^\circ}$ and $5\underline{/30^\circ}$; (*b*) $24.2\overline{\backslash 140^\circ}$ and $16.7\underline{/34^\circ}$; (*c*) $110\underline{/120^\circ}$ and $8\underline{/84^\circ}$.

Problem 2-23: Multiply (*a*) $9 - j\,4$ and $6 + j\,7$; (*b*) $- 8 + j\,3$ and $6 - j\,5$; (*c*) $- 14 - j\,11$ and $12 - j\,16$.

11. Division of Vectors. The quotient of two polar vectors equals the quotient of their moduli and the difference of their arguments.

Example 2-12: Divide $240\underline{/120^\circ}$ by $20\underline{/40^\circ}$.

Solution. $$\frac{240\underline{/120^\circ}}{20\underline{/40^\circ}} = 12\underline{/120^\circ - 40^\circ} = 12\underline{/80^\circ} \qquad \textit{Ans.}$$

The quotient of two rectangular vectors is determined by the principle of rationalization in algebra as illustrated in the following case.

Example 2-13: Divide $36 + j\,12$ by $8 - j\,4$.

Solution. Write the problem in fractional form:

$$\frac{36 + j\,12}{8 - j\,4}$$

Rationalize by multiplying both numerator and denominator of the fraction by the denominator with the sign preceding the second term changed, i.e., by the conjugate of the denominator. In this case the conjugate of the denominator is $8 + j\,4$. Then,

$$\frac{36 + j\,12}{8 - j\,4} \cdot \frac{8 + j\,4}{8 + j\,4} = \frac{288 + j\,240 + j^2\,48}{64 - j^2\,16}$$

Substituting for $+ j^2\,48$ its value $- 48$, and for $- j^2\,16$ its value $+ 16$ and simplifying, we have

$$\frac{240 + j\,240}{80} = 3 + j\,3 \qquad \textit{Ans.}$$

Example 2-14: Resistance and inductance are connected in series across a 120-volt a-c source. The current in the circuit lags the voltage by 36° 52′

and equals 5 amp. Determine the impedance of the circuit. *Note:* The a-c series circuit will be treated in more detail in Chapter III. This example, together with problem 2-26, is offered to illustrate the need for multiplication and division of vectors.

Solution. Impedance equals voltage divided by current. With current along the axis of reals, the voltage will lie in the first quadrant 36° 52′ ahead of the current.

$$\begin{aligned} Z/\underline{\theta} &= \frac{E/\underline{\theta}}{I/\underline{0^\circ}} = \frac{120/\underline{36^\circ\,52'}}{5/\underline{0^\circ}} = 24/\underline{36^\circ\,52' - 0^\circ} \\ &= 24/\underline{36^\circ\,52'} \qquad \textit{Ans.} \end{aligned}$$

Alternate Solution by Rectangular Vectors. Expressing the polar vector $E/\underline{\theta}$ in rectangular coordinates, we have,

$$\begin{aligned} 120/\underline{36^\circ\,52'} &= 120\,(\cos 36^\circ\,52' + j\sin 36^\circ\,52') \\ &= 96 + j\,72 \text{ volts} \\ \frac{96 + j\,72}{5 + j\,0} &= 19.2 + j\,14.4 \text{ ohms} \\ Z/\underline{\theta} &= \sqrt{19.2^2 + 14.4^2}/\underline{\tan^{-1} 14.4/19.2} \\ &= 24/\underline{36^\circ\,52'} \text{ ohms} \qquad \textit{Ans.} \end{aligned}$$

Impedance although expressed as a vector quantity is actually not a vector, but a complex operator which resolves the voltage into its resistive and reactive components. With current plotted on the axis of reals the voltage expressed as a rectangular vector would be:

$$ZI = RI \pm j\,(XI) \tag{2-6}$$

Dividing by the current, we have

$$Z = R \pm j\,X \tag{2-7}$$

The sign of the operator j which precedes the reactance or reactive voltage is positive when the circuit is inductive, and negative when the circuit is capacitive.

Problem 2-24: Divide (*a*) $110\backslash\overline{120^\circ}$ by $4\backslash\overline{15^\circ}$; (*b*) $440/\underline{30^\circ}$ by $20\backslash\overline{16^\circ}$.

Problem 2-25: Divide (*a*) $28 + j\,14$ by $8 - j\,6$; (*b*) $-22 - j\,12$ by $-4 + j\,2$.

Problem 2-26: The current in an a-c series circuit leads the voltage by 22°, and has a magnitude of 2.68 amp with 240 volts impressed across the circuit. Determine the impedance, resistance, and reactance of the circuit.

12. The Reciprocals of Vectors. The reciprocal of a polar vector is obtained by assuming a numerator of one along the axis of reals and dividing by the vector whose reciprocal is required.

Example 2-15: Determine the reciprocal of $8/\underline{50^\circ}$.

Solution. $$\frac{1/\underline{0^\circ}}{8/\underline{50^\circ}} = 0.125/\underline{0^\circ - 50^\circ} = 0.125\backslash\overline{50^\circ} \qquad \textit{Ans.}$$

The reciprocal of a rectangular vector is obtained by dividing the vector into one.

Example 2-16: Determine the reciprocal of $5 - j\,4$.

Solution. Rationalizing, we have

$$\frac{1}{5 - j\,4} \cdot \frac{5 + j\,4}{5 + j\,4} = \frac{5 + j\,4}{25 - j^2\,16}$$
$$= \frac{5 + j\,4}{41} = 0.122 + j\,0.0976 \qquad \textit{Ans.}$$

The reciprocal of a vector is often applied in the solution of a-c parallel circuits. The pressure across the parallel circuit is assumed as one volt. The branch current will then become the reciprocal of the branch impedance. The a-c parallel circuit will be treated in more detail in Chapter III. Example 2-17, and problem 2-29, are given here merely to illustrate the application of vector reciprocals.

Example 2-17: An a-c parallel circuit contains two branches. Branch one contains 5 ohms resistance and 10 ohms capacitive reactance, while branch two consists of 2 ohms resistance and 6 ohms inductive reactance. Determine the line current per volt.

Solution.

$$\begin{aligned}
Z_1 &= 5 - j\,10 \text{ ohm} \\
\underset{\cdot}{I}_1 &= \frac{1}{Z_1} = \frac{1}{5 - j\,10} \\
&= \frac{1}{5 - j\,10} \cdot \frac{5 + j\,10}{5 + j\,10} = \frac{5 + j\,10}{125} \\
&= 0.04 + j\,0.08 \text{ amp} \\
Z_2 &= 2 + j\,6 \text{ ohm} \\
\underset{\cdot}{I}_2 &= \frac{1}{Z_2} = \frac{1}{2 + j\,6} \\
&= \frac{1}{2 + j\,6} \cdot \frac{2 - j\,6}{2 - j\,6} = \frac{2 - j\,6}{40} \\
&= 0.05 - j\,0.15 \text{ amp} \\
\underset{\cdot}{I} &= \underset{\cdot}{I}_1 + \underset{\cdot}{I}_2 = 0.04 + j\,0.08 + 0.05 - j\,0.15 \\
&= 0.09 - j\,0.07 \text{ amp} \\
I\underline{/\theta} &= \sqrt{(0.09)^2 + (-0.07)^2}\ \underline{/\tan^{-1} - 0.07/0.09} \\
I\overline{\backslash\theta} &= 0.1062\overline{\backslash 37^\circ\,54'} \text{ amp} \qquad \textit{Ans.}
\end{aligned}$$

Problem 2-27: Determine the reciprocal of (*a*) $12.4\underline{/48^\circ}$; (*b*) $6.8\overline{\backslash 34^\circ\,20'}$; (*c*) $19.3\overline{\backslash 132^\circ\,35'}$.

Problem 2-28: Determine the reciprocal of (*a*) $8 + j\,2$; (*b*) $-16 - j\,22$; (*c*) $-4.2 + j\,6.5$.

Problem 2-29: An a-c parallel circuit has $12\underline{/45^\circ}$ ohms impedance in branch one, and $6\overline{\backslash 60^\circ}$ ohms impedance in branch two. Determine the current per volt in the line.

13. Powers and Roots of Vectors. Polar vectors are used in problems involving powers and roots of vectors. A polar vector raised to the *n*th power equals the *n*th power of the modulus, and the product of *n* and the argument.

$$(I\underline{/\theta})^n = I^n\underline{/n\theta} \tag{2-8}$$

The *n*th root of a polar vector equals the *n*th root of the modulus, and the quotient of the argument and *n*.

$$\sqrt[n]{I\underline{/\theta}} = \sqrt[n]{I}\underline{/\theta/n} \tag{2-9}$$

Example 2-18: Determine the value of (*a*) $(3\underline{/50^\circ})^4$; (*b*) $\sqrt{64\underline{/150^\circ}}$.

Solution.

(*a*)
$$\begin{aligned}(3\underline{/50^\circ})^4 &= 3^4\underline{/4 \times 50^\circ}\\ &= 81\underline{/200^\circ} \qquad \textit{Ans.}\end{aligned}$$

(*b*)
$$\begin{aligned}\sqrt{64\underline{/150^\circ}} &= \sqrt{64}\underline{/150/2}\\ &= 8\underline{/75^\circ} \qquad \textit{Ans.}\end{aligned}$$

Problem 2-30: Determine the value of (*a*) $(6\underline{/25^\circ})^2$; (*b*) $(2\overline{\backslash 30^\circ})^5$; (*c*) $\sqrt{20.25\underline{/54^\circ}}$.

Problem 2-31: Determine the value of (*a*) $(8\underline{/45^\circ})^2$; (*b*) $(3\overline{\backslash 15^\circ})^4$; (*c*) $\sqrt{27\overline{\backslash 150^\circ}}$.

14. Displacement of Vectors. Article 4 explained the method of displacing a polar vector. That is, to rotate $I\underline{/\theta}$ through the angle ϕ, we have merely to add the angle ϕ to the argument. This is equivalent to multiplying $I\underline{/\theta}$ by $1\underline{/\phi}$.

$$I\underline{/\theta} \cdot 1\underline{/\phi} = I\underline{/\theta + \phi} \tag{2-10}$$

Although a more tedious computation, a rectangular vector may also be rotated. Let it be required to rotate the rectangular vector $I\,(\cos\theta + j\sin\theta)$ through the angle ϕ (Fig. 2-27). The resultant vector will be expressed in rectangular form as

$$I[\cos(\theta + \phi) + j\sin(\theta + \phi)]$$

Expanding $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$ gives

$$I[\cos\theta\cos\phi - \sin\theta\sin\phi + j\sin\theta\cos\phi + j\cos\theta\sin\phi]$$

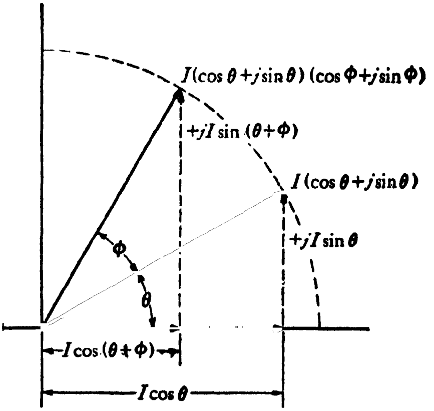


Fig. 2-27. Rotation of a Vector Expressed in Rectangular Coordinates.

Rewriting with $+ j^2 \sin\theta \sin\phi$ substituted for its equivalent $-\sin\theta \sin\phi$ to obtain a simple form of expression:

$$I[\cos\theta\cos\phi + j^2 \sin\theta \sin\phi + j\sin\theta\cos\phi + j\cos\theta\sin\phi]$$

Regrouping and simplifying:

$$I[\cos\phi\,(\cos\theta + j\sin\theta) + j\sin\phi\,(\cos\theta + j\sin\theta)]$$
$$I\,(\cos\theta + j\sin\theta)(\cos\phi + j\sin\phi) \qquad (2\text{-}11)$$

Thus to rotate a rectangular vector through an angle ϕ, simply multiply the vector by $(\cos\phi + j\sin\phi)$.

Example 2-19: Rotate $8 - j\,4$ through 120°.

Solution.

$$\begin{aligned}&(8 - j\,4)(\cos 120° + j\sin 120°)\\&= (8 - j\,4)(-0.5 + j\,0.866)\\&= -4 + j\,2 + j\,6.928 - j^2\,3.464\\&= -0.536 + j\,8.928 \qquad \textit{Ans.}\end{aligned}$$

Problem 2-32: Rotate (*a*) $1264\diagdown\overline{165°}$ through $-32°$; (*b*) $9.6 + j\,7.8$ through 150°; (*c*) $-215 - j\,144$ through $-75°$.

Problem 2-33: Rotate (*a*) $32\diagdown\overline{140°}$ through 160°; (*b*) $-46 + j\,28$ through 68°; (*c*) $4.2 - j\,3.5$ through $-24°\,16'$.

LABORATORY EXPERIMENTS

EXPERIMENT 2-1

Purpose. 1. To construct the vector diagram of an a-c series circuit from experimental data.

2. To add the voltages across the parts of an a-c series circuit graphically to obtain the impressed voltage, and to check the value obtained with the measured value.

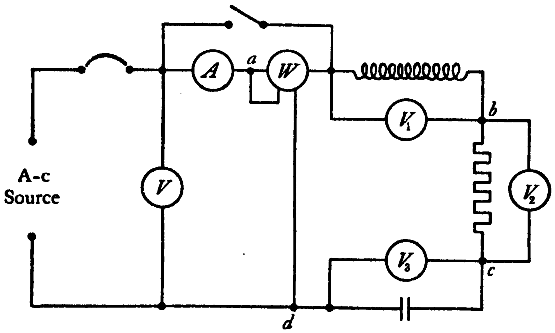


Fig. 2-28. Diagram for Experiment 2-1.

Procedure. Connect a reactor, a rheostat, and a capacitor as shown in Fig. 2-28. Adjust the rheostat so as to limit the current to a value which does not exceed the rating of the instruments or the apparatus. Check the voltages E_1, E_2, and E_3. Use a voltmeter that will read at least twice line voltage as the values of E_2 and E_3 may exceed the impressed voltage. A lower range voltmeter may be used if the first check indicates that this may be done with safety. Read and record: E, I, P, E_1, P_1, E_2, P_2, and E_3. The wattmeter potential coil will have to be shifted to *ab* to obtain P_1, and to *bc* to measure P_2. P_3 may be assumed to equal zero. This fact may be checked by shifting the wattmeter potential coil to *cd*.

Report. 1. Using the values of E, I, and P, compute θ, the phase angle between the current and the line voltage. Record the value of this angle on the data sheet.

2. Using E_2, I, and P_2, compute and record θ_2, the phase angle between current and E_2.

3. Using current as a reference, plot vectors for E_1, E_2, and E_3 to scale. Add these graphically to obtain E and θ. Record the values of E and θ thus obtained and compare them with the measured value of E and the value of θ computed from readings.

4. Submit the data sheet and the vector diagram, together with a summary of the results which were obtained.

EXPERIMENT 2-2

Purpose. 1. To construct a vector diagram of an a-c parallel circuit from experimental data.

2. To add the branch currents graphically to obtain the line current, and to check the value obtained with the measured value.

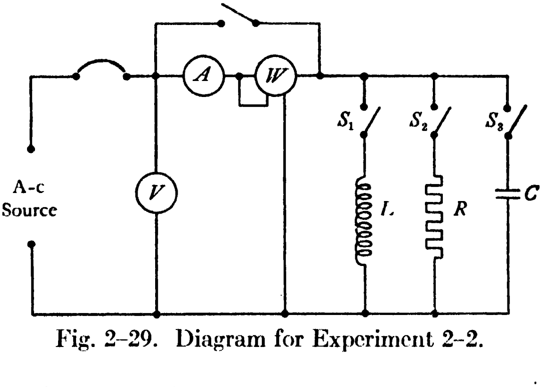


Fig. 2-29. Diagram for Experiment 2-2.

Procedure. 1. Connect a rheostat, a reactor, and a capacitor as shown in Fig. 2-29. Open switches 2 and 3, and close switch 1. Energize the circuit and read E, I, and W. Record the readings.

2. Deenergize the circuit. Open switches 1 and 3, and close switch 2. Energize the circuit, read instruments, and record data.

3. Repeat part 2, but with switch 3 closed and switches 1 and 2 open.

4. Repeat part 2 with switches 1, 2, and 3 closed.

Report. 1. Using the data of part 1, compute and record the phase angle of branch 1.

2. Using voltage as a reference, plot the three branch currents to scale. Assume I_2 to be in phase with E and I_3 to lead E by 90°. Add I_1, I_2, and I_3 graphically to obtain the line current.

3. Using data of part 4, compute and record the line power factor and the phase angle between line current and line voltage.

4. Compare the line current obtained by vector addition with the value measured in part 4 of the procedure. Also, compare the line phase angle obtained by vector addition with that computed in part 3 of the results.

5. Submit the data sheet and the vector diagram together with a summary of the results which were obtained.

CHAPTER III

SINGLE-PHASE SYSTEMS

1. Introduction. A system energized by a single alternating voltage is called a *single-phase* system. Certain characteristics of single-phase systems were presented in Chapter I, and a review of these may be in order at this time. In an a-c circuit which contains only resistance the current and voltage are *in phase*, and E divided by I equals R, the *resistance* of the circuit in ohms. If the single-phase circuit contains only inductance, the current *lags* the impressed voltage by 90°, and E divided by I equals X_L, the *inductive reactance* of the circuit in ohms. When an a-c circuit contains only capacitance, the current *leads* the impressed voltage by 90°, and E divided by I equals X_C, the *capacitive reactance* of the circuit in ohms.

In a single-phase series circuit which contains resistance and inductance, the impressed voltage equals a combination of the resistance voltage and the reactance voltage, and hence E divided by I equals a combination of R and X_L, which is known as the *impedance* of the circuit. Impedance is expressed in ohms and its symbol is Z. If the circuit contains resistance and capacitance, or resistance, inductance, and capacitance, the impressed voltage divided by the current also gives the impedance of the circuit. Hence, *the impedance of a single-phase series circuit is a combination of all quantities (expressed in ohms) which impede the flow of current in the circuit.*

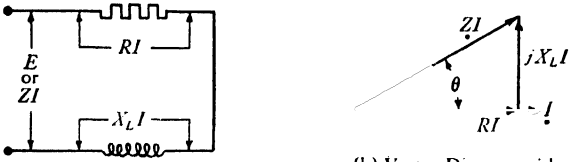
Since the resistance voltage RI, the inductive reactance voltage X_LI, and the capacitive reactance voltage X_CI differ in time phase in respect to the current, it is obvious that these voltages must be combined vectorially to obtain the impressed voltage $Z\underset{\cdot}{I}$. Similarly, R, X_L and X_C must be combined vectorially to obtain Z.

Also, in Chapter I, we have found that the power in a single-phase circuit equals the product of the current, the impressed voltage, and the cosine of the phase angle between current and voltage. *The cosine of the phase angle is called the power factor of the circuit.* When $\underset{\cdot}{I}$ lies in a clockwise direction from $\underset{\cdot}{E}$, the power factor is said to be *lagging*, and when

$\underset{\cdot}{I}$ lies in a counterclockwise direction from $\underset{\cdot}{E}$, the power factor is said to be *leading*.

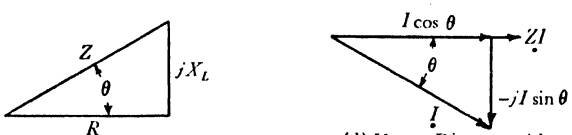
2. Resistance and Inductance in Series. When a single-phase series circuit contains only resistance and inductance, the impressed voltage $\underset{\cdot}{E}$, or $\underset{\cdot}{Z}I$, can be resolved into two components at right angles to each other. One of these components, the resistance voltage RI, is in phase with the current, and the other component, the inductive reactance voltage, X_LI, leads the current by 90°. This latter statement is equivalent to saying that $\underset{\cdot}{I}$ lags X_LI by 90°.

The various diagrams for this type of circuit are illustrated by Fig. 3-1. Since the current in a series circuit is the same in all parts of the circuit, it is customary to use the current as a reference about which to plot the voltages. This has been done in the vector diagram (Fig. 3-1*b*) by placing $\underset{\cdot}{I}$ along the axis of reals. RI is plotted in phase with $\underset{\cdot}{I}$ and X_LI is plotted 90° ahead of $\underset{\cdot}{I}$. When RI and XI are known, the impressed voltage $Z\underset{\cdot}{I}$ may be obtained by adding RI and X_LI vectorially. The value of θ, the phase angle between the impressed voltage and the current depends on the ratio of X_LI to RI, i.e., $\tan\theta$ equals X_LI divided by RI. The power factor of this type of circuit is *lagging* since $\underset{\cdot}{I}$ lies θ degrees in a clockwise direction from $\underset{\cdot}{Z}I$.



(a) Series Circuit

(b) Vector Diagram with Current as Reference



(c) Impedance Triangle

(d) Vector Diagram with Voltage as Reference

Fig. 3-1. Diagrams for an A-c Series Circuit Containing Resistance and Inductance.

Expressed in rectangular coordinates:

$$\underset{\cdot}{Z}I = RI + j\,X_LI \tag{3-1}$$

Expressing equation 3-1 in polar coordinates:

$$ZI\underline{/\theta} = \sqrt{(RI)^2 + (X_LI)^2}\;\underline{/\tan^{-1} X_LI/RI} \tag{3-2}$$

The impedance triangle (Fig. 3-1*c*) is obtained by dividing equation 3-1 by the current. This gives:

$$Z = R + jX_L \tag{3-3}$$

Which expressed in polar coordinates is:

$$Z\underline{/\theta} = \sqrt{R^2 + X_L^2}\ \underline{/\tan^{-1} X_L/R} \tag{3-4}$$

The triangle formed by Z, R, and X_L is similar to that formed by $Z\underset{\cdot}{I}$, $R\underset{\cdot}{I}$, and $X_L I$ in the vector diagram (Fig. 3-1*b*). It is evident that the impedance of this type of circuit equals the vector sum of the resistance and the inductive reactance. Also, it is evident from the impedance triangle that:

$$\theta = \sin^{-1} X_L/Z = \cos^{-1} R/Z = \tan^{-1} X_L/R \tag{3-5}$$

$$X_L = Z \sin\theta \tag{3-6}$$

$$R = Z \cos\theta \tag{3-7}$$

$$X_L = R \tan\theta \tag{3-8}$$

If we are not primarily interested in studying the relationship between the various voltages which may be considered to exist in an a-c series circuit, the vector diagram may be constructed with the impressed voltage as reference for the current as shown in Fig. 3-1*d*. If the vector diagram (Fig. 3-1*b*) is rotated clockwise through $\theta°$, $\underset{\cdot}{E}$ or $Z\underset{\cdot}{I}$ will fall along the axis of reals and $\underset{\cdot}{I}$ will lie in the fourth quadrant $\theta°$ behind $\underset{\cdot}{E}$. Hence, the vector diagram (Fig. 3-1*d*) is similar to that of Fig. 3-1*b* except that the impressed voltage instead of the current is plotted along the axis of reals.

The current may be considered as consisting of two components, one which is in phase with the voltage and the other which lags the voltage by 90°. With voltage along the axis of reals, the current for this type of circuit expressed in rectangular coordinates is:

$$\underset{\cdot}{I} \doteqdot I(\cos\theta - j\sin\theta) \tag{3-9}$$

$I\cos\theta$ is referred to as the *in-phase* current or the *resistive component* of current, while $I\sin\theta$ is called the *quadrature* current or *reactive component* of current. Attention is called to the sign preceding the operator j in equations 3-1, 3-3, and 3-9. With I as reference, the operator j for $X_L I$ in the vector diagram (Fig. 3-1*b*) and for X_L in the impedance triangle (Fig. 3-1*c*) is positive. With E as reference, as shown in the vector diagram (Fig. 3-1*d*) the sign of the j operator which precedes the quadrature or vertical component of current is negative. Writing equation 3-9 in polar coordinates we have

$$I\overline{\diagdown\theta} = \sqrt{(I\cos\theta)^2 + (I\sin\theta)^2}\ \overline{\diagdown\tan^{-1} I\sin\theta/I\cos\theta} \tag{3-10}$$

Example 3-1: An effective resistance of 30 ohms is connected in series with 50 ohms inductive reactance across a 230-volt a-c source. Determine

(*a*) the impedance; (*b*) the current; (*c*) the *RI* voltage; and (*d*) the X_LI voltage.

Solution.

(*a*) $Z = 30 + j\,50$

$$Z\underline{/\theta} = \sqrt{30^2 + 50^2}\,\underline{/\tan^{-1} 50/30}$$

$$= 58.3\underline{/59^\circ} \text{ ohms} \qquad \textit{Ans.}$$

(*b*)
$$\dot{I} = \dot{E}/Z = \frac{230}{30 + j\,50}$$

$$= \frac{230}{30 + j\,50} \cdot \frac{30 - j\,50}{30 - j\,50} = \frac{6900 - j\,11{,}500}{3400}$$

$$= 2.03 - j\,3.38$$

$$I\underline{/\theta} = \sqrt{(2.03)^2 + (-\,3.38)^2}\,\underline{/\tan^{-1} -\,3.38/2.03}$$

$$= 3.95\overline{\backslash 59^\circ} \text{ amp} \qquad \textit{Ans.}$$

(*b*) **Alternate Solution for Part b**

$$I\overline{\backslash\theta} = \frac{E}{Z\underline{/\theta}} = \frac{230}{58.3\underline{/59^\circ}}$$

$$= 3.95\underline{/0^\circ - 59^\circ} = 3.95\overline{\backslash 59^\circ} \text{ amp} \qquad \textit{Ans.}$$

(*c*) $RI = 30 \times 3.95 = 118.5$ volts *Ans.*

(*d*) $X_LI = 50 \times 3.95 = 197.5$ volts *Ans.*

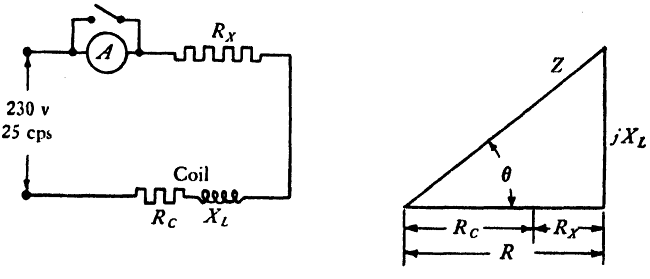


Fig. 3-2. Diagrams for Example 3-2.

Example 3-2: An air-core coil having a resistance of 40 ohms and inductance of 0.318 henry is connected in series with a noninductive resistance across a 240-volt, 25-cycle source. (*a*) What value of noninductive resistance is required to limit the current in the circuit to 3 amp? (*b*) Determine the impedance and current of the circuit.

Solution. (*a*) See Fig. 3-2. Since noninductive resistance is added, the reactance of the circuit will not be changed.

$$X_L = \omega L = 6.28 \times 25 \times 0.318 = 50 \text{ ohms}$$

The impedance of the circuit will be

$$Z\underline{/\theta} = \frac{E}{I\overline{\backslash\theta}} = \frac{240}{3\overline{\backslash\theta}} = 80\underline{/\theta} \text{ ohms}$$

In magnitude

$$\begin{aligned} Z &= \sqrt{R^2 + X_L{}^2} \\ R &= \sqrt{Z^2 - X_L{}^2} \\ R &= \sqrt{80^2 - 50^2} = 62.45 \text{ ohms} \\ R_X &= 62.45 - 40 = 22.45 \text{ ohms} \qquad \textit{Ans.} \end{aligned}$$

$$\begin{aligned} (b) \qquad \theta &= \tan^{-1} 50/62.45 = 38° \, 46' \\ Z\underline{/\theta} &= 80\underline{/38° \, 46'} \text{ ohms} \qquad \textit{Ans.} \\ I\overline{\backslash\theta} &= 3\overline{\backslash 38° \, 46'} \text{ amp} \qquad \textit{Ans.} \end{aligned}$$

Problem 3-1: An effective resistance of 40 ohms at 25 cycles and an inductance of 0.22 henry are connected in series across a 120-volt, 25-cycle source. Determine (*a*) the impedance; (*b*) the current.

Problem 3-2: A coil having 12 ohms effective resistance at 50 cycles and 0.05 henry inductance is connected in series with a second coil having 14 ohms effective resistance and 0.03 henry inductance across a 220-volt, 50-cycle source. Determine (*a*) the impedance; (*b*) the current.

Problem 3-3: A resistor which is noninductively wound is connected in series with a reactor of negligible resistance across a 230-volt, 60-cycle source. The voltage across the resistor is 80 volts. Determine the voltage across the reactor.

Problem 3-4: A coil with 16 ohms effective resistance and 12 ohms inductive reactance at 25 cycles is connected in series with a second resistance across a 180-volt 25-cycle source. (*a*) Determine the value of the second resistance if the current is to be limited to 6 amp. (*b*) Find the impedance of the series circuit.

Problem 3-5: A coil with an air core has an impedance of $48\underline{/26°}$ ohms when tested on a 25-cycle source. What is its impedance at 60 cycles?

3. Resistance and Capacitance in Series. In an a-c series circuit which contains resistance and capacitance the impressed voltage can be resolved into two components, RI which is in phase with the current, and X_CI which lags the current by 90°. The vector diagram (Fig. 3-3*b*) shows this relationship. Current is used as reference and is plotted along the axis of reals. RI is drawn in phase with $\underset{\bullet}{I}$ and X_CI is drawn 90° behind the current, i.e., current leads the capacitive reactance voltage by 90°. The vector sum of RI and X_CI gives the impressed voltage $Z\underset{\bullet}{I}$. The value of the phase angle between the impressed voltage and the current depends on the ratio of X_CI to RI, i.e., tan θ equals X_CI divided by RI. The power factor of this type of circuit is *leading* since $\underset{\bullet}{I}$ lies θ degrees in a counterclockwise direction from $Z\underset{\bullet}{I}$. Expressed in rectangular coordinates:

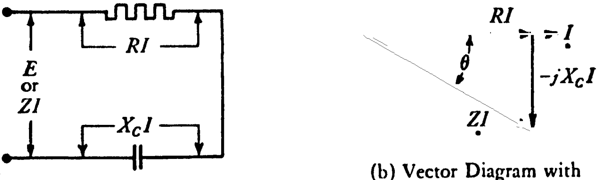
$$Z\underset{\bullet}{I} = RI - j\,X_CI \tag{3-11}$$

Changing equation 3-11 to polar form we have

$$ZI\overline{\backslash\theta} = \sqrt{(RI)^2 + (X_CI)^2}\;\overline{\backslash\tan^{-1} X_CI/RI} \tag{3-12}$$

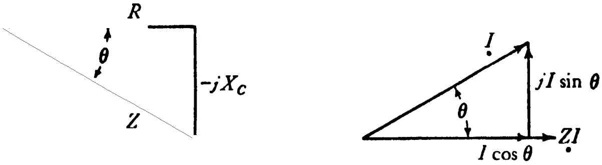
The impedance triangle (Fig. 3-3*c*) is similar to the triangle formed by ZI, RI, and X_CI in the vector diagram (Fig. 3-3*b*) and is obtained by dividing equation 3-11 by the current. Thus

$$Z = R - jX_C \tag{3-13}$$



(a) Series Circuit

(b) Vector Diagram with Current as Reference



(c) Impedance Triangle

(d) Vector Diagram with Voltage as Reference

Fig. 3-3. Diagrams for an A-c Series Circuit Containing Resistance and Capacitance.

Which expressed in polar coordinates is

$$Z\overline{\backslash\theta} = \sqrt{R^2 + X_C^2}\,\overline{\backslash\tan^{-1} X_C/R} \tag{3-14}$$

It is evident from the impedance triangle that the impedance of this type of circuit equals the vector sum of the resistance and the capacitive reactance. Also, it is evident from the impedance triangle that:

$$\theta = \sin^{-1} X_C/Z = \cos^{-1} R/Z = \tan^{-1} X_C/R \tag{3-15}$$

$$X_C = Z \sin \theta \tag{3-16}$$

$$R = Z \cos \theta \tag{3-17}$$

$$X_C = R \tan \theta \tag{3-18}$$

The vector diagram (Fig. 3-3*d*) is obtained by rotating the diagram (Fig. 3-3*b*) counterclockwise through θ degrees. The impressed voltage E or ZI now lies along the axis of reals and I lies in the first quadrant θ degrees ahead of E. With E on the axis of reals the current may be considered as consisting of an in-phase component and a quadrature component, and may be expressed in rectangular coordinates as

$$I = I(\cos \theta + j \sin \theta) \tag{3-19}$$

Attention is called to the sign preceding the operator j in equations 3-11, 3-13, and 3-19. When writing voltage and impedance with $\underset{\cdot}{I}$ on the axis of reals $-j$ is used, but when writing current with $\underset{\cdot}{E}$ on the reference axis $+j$ is used. Writing equation 3-19 in polar form, we have

$$I\underline{/\theta} = \sqrt{(I\cos\theta)^2 + (I\sin\theta)^2}\ \underline{/\tan^{-1} I\sin\theta/I\cos\theta} \qquad (3\text{-}20)$$

Example 3-3. A resistance of 40 ohms is connected in series with 50-mfd capacitance across a 220-volt, 60-cycle source. Determine (*a*) the impedance; (*b*) the current.

Solution.

$$(a)\quad X_C = \frac{10^6}{\omega C} = \frac{10^6}{6.28 \times 60 \times 50} = 53.1 \text{ ohm}$$

$$Z = 40 - j\,53.1 \text{ ohms}$$

$$Z\underline{/\theta} = \sqrt{40^2 + (-53.1)^2}\ \underline{/\tan^{-1} - 53.1/40}$$

$$= 66{,}5\overline{\backslash 53^\circ} \text{ ohm} \qquad \textit{Ans.}$$

$$(b)\quad \underset{\cdot}{I} = \frac{\underset{\cdot}{E}}{\underset{\cdot}{Z}} = \frac{220}{40 - j\,53.1}$$

$$= \frac{220}{40 - j\,53.1} \cdot \frac{40 + j\,53.1}{40 + j\,53.1} = \frac{8800 + j\,11{,}688}{4419.6}$$

$$= 1.99 + j\,2.64 \text{ amp}$$

$$I\underline{/\theta} = \sqrt{1.99^2 + 2.64^2}\ \underline{/\tan^{-1} 2.64/1.99}$$

$$= 3.31\underline{/53^\circ} \text{ amp} \qquad \textit{Ans.}$$

(*b*) **Alternate Solution for Current**

$$I\underline{/\theta} = \frac{E}{Z\overline{\backslash\theta}} = \frac{220}{66.5\overline{\backslash 53^\circ}}$$

$$= 3.31\underline{/0 - (-53^\circ)} = 3.31\underline{/53^\circ} \text{ amp} \qquad \textit{Ans.}$$

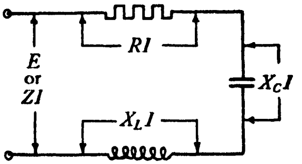
Problem 3-6: An a-c series circuit contains 36 ohms noninductive resistance and 32-mfd capacitance. What current will this circuit draw when connected across a 440-volt, 50-cycle source?

Problem 3-7: A resistor and capacitor are connected in series across a 120-volt, 25-cycle source and draw 0.54 amp. The voltage across the resistor is 96 volts. Determine (*a*) the voltage across the capacitor; (*b*) the capacitance in mfd; (*c*) the impedance expressed in polar form.

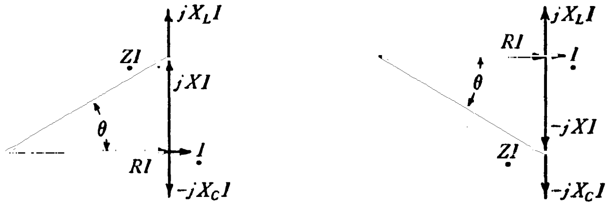
Problem 3-8: The impedance of an a-c series circuit containing a noninductive resistor and a capacitor is measured on a 60-cycle source and found to be $72\overline{\backslash 30^\circ}$ ohms. What will be the impedance of this circuit at 40 cycles?

Problem 3-9: A resistance of 30 ohms, and a capacitance of 33.2 mfd are connected in series across a 230-volt, 60-cycle source. It is desired to double the current in this circuit by adding capacitance. Determine the additional mfd capacitance required.

4. Resistance, Inductance, and Capacitance in Series. When an a-c circuit consists of resistance, inductance, and capacitance connected in

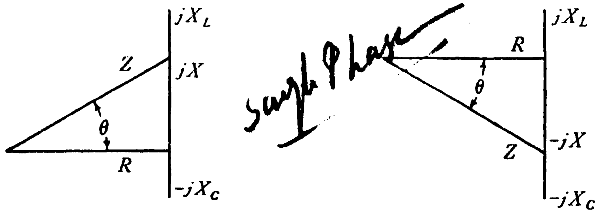


(a) Series Circuit



(b) Vector Diagram, $X_L > X_C$, Current as Reference

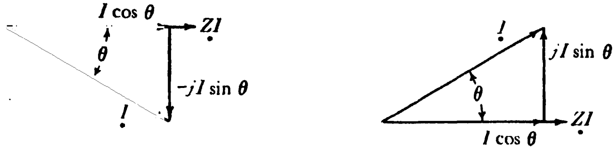
(c) Vector Diagram, $X_L < X_C$, Current as Reference



single phase

(d) Impedance Diagram, $X_L > X_C$

(e) Impedance Diagram, $X_L < X_C$



(f) Vector Diagram, $X_L > X_C$, Voltage as Reference

(g) Vector Diagram, $X_L < X_C$, Voltage as Reference

Fig. 3-4. Diagrams for an A-c Series Circuit Containing Resistance, Inductance, and Capacitance.

series (Fig. 3-4) the components of the impressed voltage ZI are the resistance voltage RI and the total reactance voltage XI. The total reactance voltage,

$$jXI = j(X_LI - X_CI) \tag{3-21}$$

The impressed voltage expressed in rectangular coordinates is

$$\underset{\cdot}{Z}I = RI + j\,(X_LI - X_CI) \tag{3-22}$$

and expressed in polar coordinates is

$$ZI\underline{/\theta} = \sqrt{RI^2 + (X_LI - X_CI)^2}\;\underline{/\tan^{-1}(X_LI - X_CI)/RI} \tag{3-23}$$

The vector diagram (Fig. 3-4*b*) is for a circuit in which X_LI is greater than X_CI. $\underset{\cdot}{I}$ is plotted along the axis of reals, RI is in phase with $\underset{\cdot}{I}$, and XI is plotted 90° ahead of $\underset{\cdot}{I}$. Since X_LI is greater than X_CI, the total reactance voltage XI is treated as an inductive reactance voltage. The power factor in this case is lagging. Fig. 3-4*c* is the vector diagram for a circuit in which X_LI is less than X_CI. When this condition exists XI lies 90° behind $\underset{\cdot}{I}$ and is treated as a capacitive reactance voltage. A leading power factor exists in the circuit when $X_LI < X_CI$.

The impedance diagram (Fig. 3-4*d*) is for a series circuit in which $X_L > X_C$, and is obtained by dividing the voltages of diagram (Fig. 3-4*b*) by the current. Fig. 3-4*e* is the impedance diagram for a series circuit in which $X_L < X_C$, and is obtained by dividing the voltages of diagram (Fig. 3-4*c*) by the current. The total reactance,

$$jX = j\,(X_L - X_C) \tag{3-24}$$

It is obvious that the impedance is the vector sum of R and X regardless of whether X is inductive or capacitive in its effect. Expressed in rectangular coordinates,

$$Z = R + jX \tag{3-25}$$

Expressed in polar coordinates,

$$Z\underline{/\theta} = \sqrt{R^2 + X^2}\;\underline{/\tan^{-1} X/R} \tag{3-26}$$

The vector diagram (Fig. 3-4*f*) illustrates the position of the current in a series circuit containing R, L, and C when $X_L > X_C$. Fig. 3-4*f* is obtained by rotating diagram (Fig. 3-4*b*) clockwise through θ degrees. The voltage $\underset{\cdot}{Z}I$ now lies along the axis of reals and $\underset{\cdot}{I}$ lags the voltage by θ degrees. The current in this type of circuit is inductive and equations 3-9 and 3-10 apply.

The vector diagram (Fig. 3-4*g*) illustrates the position of the current when $X_L < X_C$, and is obtained by rotating Fig. 3-4*c* counterclockwise through θ degrees. I in this case is a capacitive or leading current, and with voltage plotted on the axis of reals, the current equations 3-19 and 3-20 apply.

Example 3-4: An a-c series circuit has a resistance of 10 ohms, an inductive reactance of 40 ohms, and a capacitive reactance of 60 ohms when connected across a 220-volt, 60-cycle source. Determine (*a*) the impedance of the circuit expressed in polar coordinates; (*b*) the current expressed in polar coordinates; (*c*) the RI, X_LI, and X_CI voltages.

Solution.

$$
\begin{aligned}
&(a) & Z &= R + j\,(X_L - X_C) = 10 + j\,(40 - 60) & \\
& & &= 10 - j\,20 \text{ ohms} & \textit{Ans.}\\
& & Z\underline{/\theta} &= \sqrt{10^2 + (-20)^2}\;\underline{/\tan^{-1} - 20/10} & \\
& & &= 22.37\backslash\overline{63^\circ 26'} \text{ ohms} & \textit{Ans.}\\
&(b) & I\underline{/\theta} &= \frac{E}{Z\backslash\overline{\theta}} = \frac{220}{22.37}\;\underline{/0 - (-63^\circ 26')} & \\
& & &= 9.85\underline{/63^\circ 26'} \text{ amp} & \textit{Ans.}\\
&(c) & RI &= 10 \times 9.85 = 98.5 \text{ volts} & \textit{Ans.}\\
& & X_LI &= 40 \times 9.85 = 394 \text{ volts} & \textit{Ans.}\\
& & X_CI &= 60 \times 9.85 = 591 \text{ volts} & \textit{Ans.}
\end{aligned}
$$

Attention is called to the fact that the X_LI and X_CI voltages as determined in example 3-4 exceed the impressed voltage in magnitude. Thus, in a circuit of this type voltages high enough to be dangerous to persons and equipment may be developed. The X_LI and X_CI voltages may be held to relatively low values by connecting adequate resistance in series.

Problem 3-10: An effective resistance of 25 ohms at 60 cycles, 0.2-henry inductance, and 60-mfd capacitance are connected in series across a 220-volt, 60-cycle source. Determine (*a*) the magnitude and phase angle of the current; (*b*) the RI, X_LI, and X_CI voltages.

Problem 3-11: Determine the voltage required to cause 3.5 amp to flow in an a-c series circuit containing 18 ohms resistance, 9 ohms inductive reactance, and 22 ohms capacitive reactance.

Problem 3-12: An a-c series circuit containing noninductive resistance and inductance has an impedance of $68\underline{/50^\circ}$ ohms at 60 cycles. A second series circuit containing noninductive resistance and capacitance has an impedance of $36\backslash\overline{25^\circ}$ ohms at 60 cycles. Determine the magnitude and phase angle of the current should these two circuits be connected in series across a 220-volt, 25-cycle source.

5. Resonance in A-c Series Circuits. An a-c series circuit is said to be *in resonance* when the inductive reactance X_L equals the capacitive reactance X_C. Under this condition the total reactance X equals zero and the impedance equals the resistance of the circuit. The phase angle between the current and voltage in a resonant circuit is zero and the power factor is unity. It is obvious then that the current in an a-c series circuit reaches its greatest magnitude when the circuit is in resonance.

Inductive reactance varies directly with frequency, while capacitive reactance varies inversely with frequency. These relationships are expressed by the equations,

$$
\begin{aligned}
X_L &= \omega L = 2\pi f L\\
X_C &= 1/\omega C = 1/2\pi f C
\end{aligned}
$$

At zero frequency X_L is zero and X_C infinite in value. As the frequency increases, X_L also increases, but X_C decreases. When the frequency becomes infinite, X_L is infinite and X_C is zero. Hence, in a series circuit containing inductance and capacitance, X_L will equal X_C at some value of frequency. This value of frequency can be determined by placing X_L equal to X_C and solving for f.

$$\begin{aligned} 2\pi fL &= 1/2\pi fC \\ 4\pi^2f^2LC &= 1 \\ f^2 &= 1/4\pi^2LC \\ f &= 1/2\pi\sqrt{LC} \qquad (3\text{-}27) \end{aligned}$$

where L is in henries and C is in farads. If C is expressed in microfarads, equation 3-24 becomes

$$f = 10^3/2\pi\sqrt{LC} \qquad (3\text{-}28)$$

Example 3-5: An inductance of 0.06 henry, 2.4-mfd capacitance, and 350 ohms of noninductive resistance are connected in series. Determine resonant frequency for this circuit.

Solution.

$$\begin{aligned} f &= 10^3/2\pi\sqrt{LC} = 1000/6.28\sqrt{0.06\times 2.4} \\ &= 420 \text{ cps} \qquad \textit{Ans.} \end{aligned}$$

If the product of L and C is held constant, it is obvious, from equation 3-27, that resonant frequency would also be constant. It is possible then to produce resonance at a definite frequency with an infinite number of combinations of L and C, provided the product LC remains constant. The product LC necessary for resonance in a series circuit at 60 cycles per second can be determined as follows: solve equation 3-28 for LC,

$$\begin{aligned} \sqrt{LC} &= \frac{10^3}{2\pi f} = \frac{10^3}{(2\pi)(60)} = 2.655 \\ LC &= 7.04 \end{aligned}$$

where L is in henrys and C is in microfarads.

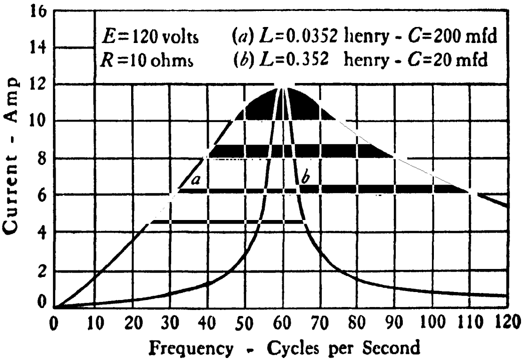
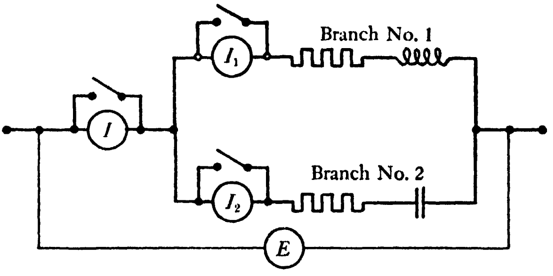


Fig. 3–5. Series Circuit Resonant Curves.

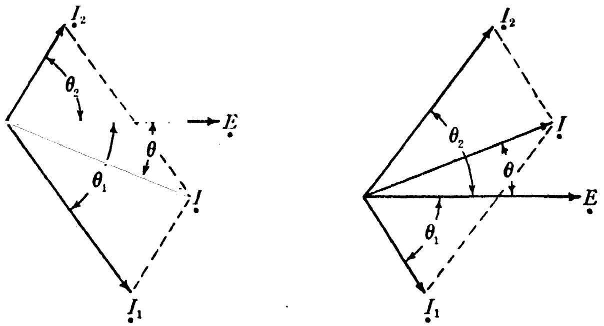
Curves with frequency as abscissa and current as ordinate are shown in Fig. 3-5 for two circuits. The voltage applied to each circuit is held constant at 120 volts. Curve (*a*) is for an a-c series circuit with 10 ohms resistance, 0.0352-henry inductance, and 200-mfd capacitance; while curve (*b*) is for a series circuit having 10 ohms resistance, 0.352-henry inductance, and 20-mfd capacitance. The product LC equals 7.04 for each circuit, hence resonance in each takes place at 60 cycles. However, the shape of the two curves differs and attention is called to the fact that increasing inductance and decreasing capacitance sharpen the resonant curve.

Problem 3-13: A noninductive resistance of 10 ohms is connected in series with 0.25-henry inductance and 35-mfd capacitance across a 110-volt, variable frequency source. (*a*) Determine resonant frequency; (*b*) construct a data sheet with columns for: f, R, X_L, X_C, X, $Z\underline{/\theta}$, and $I\underline{/\theta}$. Compute the data for frequencies of 30, 40, 50, 60, 70, and 80 cycles per second, as well as for resonant frequency; (*c*) with frequency as abscissa, plot the magnitudes of the following as ordinates: R, X_L, X_C, Z, and I. Assume the resistance to be constant for all frequencies.

Problem 3-14: (*a*) Determine resonant frequency; (*b*) construct a data sheet; and (*c*) plot curves as requested in problem 3-13, but for a series circuit with 10 ohms resistance, 0.35-henry inductance, and 25-mfd capacitance.



(a) Parallel Circuit



(b) Vector Diagram, $I_1 \sin\theta_1 > I_2 \sin\theta_2$

(c) Vector Diagram, $I_1 \sin\theta_1 < I_2 \sin\theta_2$

Fig. 3-6. Diagrams for an A-c Parallel Circuit.

6. A-c Parallel Circuits. Fig. 3-6 shows the circuit and vector diagram for a single-phase parallel circuit containing resistance and inductance in branch one and resistance and capacitance in branch two. Since the voltage across the parallel circuit is common to the branch currents, $\underset{\cdot}{E}$ is used as the reference for $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$, and is plotted along the axis of reals. The current $\underset{\cdot}{I}_1$ lags the voltage because branch one contains resistance and inductance in series. Hence, $\underset{\cdot}{I}_1$ is plotted in the fourth quadrant θ_1 degrees behind E. The value of θ_1 can be determined by equation 3-5 if r_1 and x_1 are known, or from the equation $\theta_1 = \cos^{-1} P_1/EI_1$, if P_1, E, and I_1 are known. This latter equation is a rearranged form of the power equation $P = EI \cos \theta$. The current $\underset{\cdot}{I}_2$ leads the voltage because branch two contains resistance and capacitance, and hence $\underset{\cdot}{I}_2$ is plotted in the first quadrant θ_2 degrees ahead of $\underset{\cdot}{E}$. The value of θ_2 can be determined by equation 3-15 if r_2 and x_2 are known, or by the equation $\theta_2 = \cos^{-1} P_2/EI_2$, if P_2, E, and I_2 are known.

The line current $\underset{\cdot}{I}$ is the vector sum of the branch currents $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$, and the angle θ between the line current and the voltage is the phase angle of the parallel circuit. When the vertical component of current in the inductive branch exceeds the vertical component of current in the capacitive branch (Fig. 3-6*b*), the line current lags the voltage. When the vertical component of current in the inductive branch is less than that of the capacitive branch, the line current leads the voltage as shown in Fig. 3-6*c*.

A parallel circuit having more than two branches is treated in a similar manner. The magnitude and phase angle of each branch current is determined, and the branch currents are added vectorially to obtain the magnitude and phase angle of the line current. Example 3-6 shows how the magnitude and phase angle of the line current in a parallel circuit can be obtained by the use of vector algebra.

Example 3-6. A parallel circuit is connected across a 220-volt, 60-cycle source. Branch one contains 30 ohms resistance and 40 ohms inductive reactance, and branch two contains 50 ohms resistance and 80 ohms capacitive reactance.

Determine the magnitude and phase angle of the line current.

Solution. See diagrams, Fig. 3-6*a* and Fig. 3-6*b*.

$$
\begin{aligned}
I_1 &= \frac{E}{z_1} = \frac{E}{r_1 + j\,x_1} = \frac{220}{30 + j\,40} \\
&= \frac{220}{30 + j\,40} \cdot \frac{30 - j\,40}{30 - j\,40} = \frac{6600 - j\,8800}{2500} \\
&= 2.64 - j\,3.52 \text{ amp} \\
\underset{\cdot}{I}_2 &= \frac{E}{z_2} = \frac{E}{r_2 - j\,x_2} = \frac{220}{50 - j\,80}
\end{aligned}
$$

$$= \frac{220}{50 - j\,80} \cdot \frac{50 + j\,80}{50 + j\,80} = \frac{11{,}000 + j\,17{,}600}{8900}$$
$$= 1.237 + j\,1.98 \text{ amp}$$
$$\underset{\cdot}{I} = \underset{\cdot}{I}_1 + \underset{\cdot}{I}_2 = 2.64 - j\,.352 + 1.237 + j\,1.98$$
$$= 3.88 - j\,1.54 \text{ amp} \qquad \textit{Ans.}$$
$$I\underline{/\theta} = \sqrt{3.88^2 + 1.54^2}\,\underline{/\tan^{-1} - 1.54/3.88}$$
$$= 4.17\overline{\backslash 21^\circ\, 40'} \text{ amp} \qquad \textit{Ans.}$$

7. Series Equivalent Impedance of a Parallel Circuit. A series impedance which draws current of the same magnitude and phase angle as that drawn by a parallel circuit when placed across the same single-phase source is called the *series equivalent impedance* of the parallel circuit. Series equivalent impedance is usually referred to simply as the impedance of the parallel circuit, and can be determined in the following manner:

1. Solve for and express in rectangular form the current in each branch circuit.
2. Add the branch currents vectorially to obtain the line current.
3. Divide the voltage by the line current to determine the series equivalent impedance.

When the series equivalent impedance of a single-phase parallel circuit is expressed in rectangular form, its first term is the *series equivalent resistance* of the circuit, and its second term is the *series equivalent reactance.*

Example 3-7: Determine: (*a*) the series equivalent impedance; (*b*) the series equivalent resistance; and (*c*) the series equivalent reactance of the parallel circuit of example 3-6.

Solution. (*a*) From the solution of example 3-6, the line current expressed in rectangular coordinates is

$$\underset{\cdot}{I} = 3.88 - j\,1.54 \text{ amp}$$
$$Z = \frac{E}{I} = \frac{220}{3.88 - j\,1.54}$$
$$= \frac{220}{3.88 - j\,1.54} \cdot \frac{3.88 + j\,1.54}{3.88 + j\,1.54} = \frac{854 + j\,339}{17.37}$$
$$= 49.1 + j\,19.5 \text{ ohms}$$
$$Z\underline{/\theta} = \sqrt{49.1^2 + 19.5^2}\,\underline{/\tan^{-1} 19.5/49.1}$$
$$= 52.74\underline{/21^\circ\, 40'} \text{ ohms} \qquad \textit{Ans.}$$

(*b*) The series equivalent resistance = 49.1 ohms *Ans.*

(*c*) The series equivalent reactance = 19.5 ohms inductive *Ans.*

Problem 3-15: A parallel circuit of three branches is connected across a 440-volt, 25-cycle source. Branch one has 30 ohms resistance, 40 ohms inductive reactance, and 80 ohms capacitive reactance. Branch two contains 50 ohms resistance and

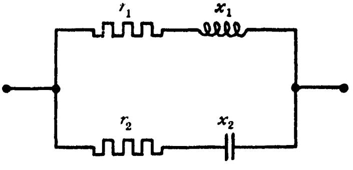
60 ohms inductive reactance. Branch three contains 60 ohms resistance, 90 ohms inductive reactance, and 75 ohms capacitive reactance. Determine the magnitude and phase angle of the line current.

Problem 3-16: Determine (*a*) the series equivalent impedance; (*b*) the series equivalent resistance; and (*c*) the series equivalent reactance of the parallel circuit of problem 3-15.

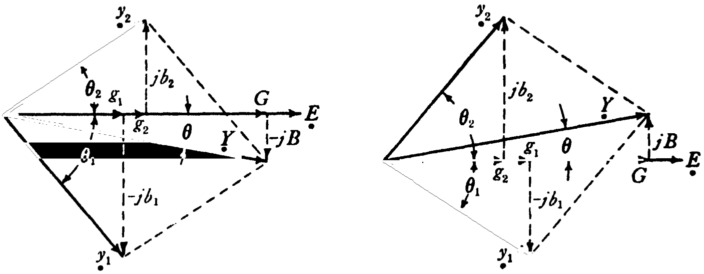
8. Admittance, Conductance, and Susceptance.

When the values of the resistances and the reactances in a single-phase parallel circuit are known, the admittance method is frequently used to determine the series equivalent impedance of the circuit. This method differs in principle from that given in the previous two articles only in that unit voltage is assumed across the branches of the parallel circuit.

The parallel circuit (Fig. 3-7) consists of two branch circuits with resistance and inductance in one, and resistance and capacitance in the other. If we consider the impressed voltage to equal unity and to lie along the axis of reals, the current $\underset{\cdot}{I}_1$, which in this case is the current per volt in branch one, can be computed as follows:



(a) Parallel Circuit



(b) Vector Diagram, $b_1 > b_2$ (c) Vector Diagram, $b_1 < b_2$

Fig. 3-7. Diagrams for Admittance Method of Solving A-c Parallel Circuits.

$$\underset{\cdot}{I}_1 = \frac{1}{r_1 + j\,x_1} = \frac{1}{r_1 + j\,x_1} \cdot \frac{r_1 - j\,x_1}{r_1 - j\,x_1}$$

$$= \frac{r_1 - j\,x_1}{r_1^2 + x_1^2} = \frac{r_1}{r_1^2 + x_1^2} - j\,\frac{x_1}{r_1^2 + x_1^2}$$

The current per volt in an a-c circuit is called the *admittance* of the circuit and is represented by the symbol y. Thus, y_1 could be substituted for $\underset{\cdot}{I}_1$ in the above equation and the admittance of the first branch becomes

$$\underset{\cdot}{y}_1 = \frac{r_1}{r_1^2 + x_1^2} - j\frac{x_1}{r_1^2 + x_1^2} \tag{3-29}$$

The horizontal component of the current per volt is the *conductance* of the circuit and is represented by the symbol g. The vertical component of the current per volt is the *susceptance* and is represented by b. It is evident from equation 3-29, that

$$g_1 = \frac{r_1}{r_1^2 + x_1^2} \tag{3-30}$$

and

$$b_1 = \frac{x_1}{r_1^2 + x_2^2} \tag{3-31}$$

Using symbols to represent conductance and susceptance, equation 3-29 can be written,

$$y_1 = g_1 - j\,b_1 \tag{3-32}$$

The equation for the admittance of the second branch is

$$y_2 = g_2 + j\,b_2$$

The sign indicating the direction of the susceptance is positive for the second branch, since the branch current leads the voltage.

The vector diagram (Fig. 3-7*b*) is drawn with voltage along the axis of reals. The conductances g_1 and g_2 are plotted in phase with $\underset{\cdot}{E}$. The susceptance b_1 is plotted 90° behind $\underset{\cdot}{E}$, and b_2 is plotted 90° ahead of $\underset{\cdot}{E}$. The admittance y_1 is the vector sum of g_1 and b_1, and y_2 is the vector sum of g_2 and b_2. Capital G is used to represent the total conductance of the parallel circuit and capital B is used as the symbol for the total susceptance. It is evident from the vector diagram that

$$G = g_1 + g_2$$

and

$$B = b_1 + b_2$$

The values of B, b_1, and b_2 may be positive or negative, depending on the capacitive or inductive character of the circuit for which they are used. The symbol Y represents the total admittance, or the total current per volt, in the parallel circuit. Y equals the vector sum of y_1 and y_2; that is:

$$\begin{aligned}\underset{\cdot}{Y} &= g_1 - j\,b_1 + g_2 + j\,b_2\\ &= (g_1 + g_2) + j\,(-\,b_1 + b_2)\\ &= G + j\,B\end{aligned} \tag{3-33}$$

Expressed in polar coordinates:

$$Y\underline{/\theta} = \sqrt{G^2 + B^2}\,\underline{/\tan^{-1} B/G} \tag{3-34}$$

If the unit voltage which is assumed to lie along the axis of reals is divided by the total current per volt, the result is the series equivalent impedance of the parallel circuit. Thus,

$$Z\overline{\backslash\theta} = 1/Y\underline{/\theta} \tag{3-35}$$

Equation 3-35 may be transposed to read,

$$Y\underline{/\theta} = 1/Z\overline{\backslash\theta} \tag{3-36}$$

The following equations are evident from the vector diagrams of Fig. 3-7:

$$\theta = \cos^{-1} G/Y = \sin^{-1} B/Y = \tan^{-1} B/G \tag{3-37}$$

Since the series equivalent resistance of a parallel circuit is the horizontal component of its series equivalent impedance,

$$R = Z\cos\theta = ZG/Y = G/Y^2 \tag{3-38}$$

And since the series equivalent reactance of a parallel circuit is the vertical component of its series equivalent impedance,

$$X = Z\sin\theta = ZB/Y = B/Y^2 \tag{3-39}$$

If a voltage E having a value other than unity is impressed on a parallel circuit, the current at that voltage can be obtained by multiplying the magnitude of the voltage by the admittance. That is,

$$I\underline{/\theta} = EY\underline{/\theta} \tag{3-40}$$

$$\underset{\cdot}{I} = \underset{\cdot}{E}\,(G + jB) \tag{3-41}$$

Admittance, conductance, and susceptance are expressed in *mho* or reciprocal ohms. Attention is called to the use of lower-case letters for the branch circuits and the use of capital letters for the line values. This method of indicating parallel circuit quantities is recommended in order to avoid confusion.

Example 3-8: An a-c parallel circuit contains 30 ohms resistance and 40 ohms inductive reactance in series in branch one. Branch two consists of 50 ohms resistance, 75 ohms inductive reactance, and 125 ohms capacitive reactance connected in series. Determine: (*a*) the conductance; (*b*) the susceptance; (*c*) the admittance; (*d*) the equivalent impedance; (*e*) the equivalent resistance; (*f*) the equivalent reactance.

Solution.

(*a*) $$y_1 = \frac{1}{r_1 + jx_1} = \frac{1}{30 + j40}$$

$$\frac{1}{30 + j40} \cdot \frac{30 - j40}{30 - j40} = \frac{30 - j40}{2500}$$

$$= 0.012 - j0.016$$

$$y_2 = \frac{1}{r_2 + j\,(x_L - x_C)} = \frac{1}{50 - j\,50}$$

$$= \frac{1}{50 - j\,50} \cdot \frac{50 + j\,50}{50 + j\,50} = \frac{50 + j\,50}{5000}$$

$$= 0.01 + j\,0.01$$

$$G = g_1 + g_2 = 0.012 + 0.01 = 0.022 \text{ mho} \qquad \textit{Ans.}$$

$$(b)\quad j\,B = -\,j\,0.016 + j\,0.01 = -\,j\,0.006 \text{ mho} \qquad \textit{Ans.}$$

$$(c)\quad Y = G + j\,B = 0.022 - j\,0.006$$

$$Y\underline{/\theta} = \sqrt{0.022^2 + 0.006^2}\;\underline{/\tan^{-1} - 0.006/0.022}$$

$$= 0.0228\overline{\backslash 15^\circ\, 16'} \text{ mho} \qquad \textit{Ans.}$$

$$(d)\quad Z\underline{/\theta} = \frac{1}{Y\overline{\backslash\theta}} = \frac{1}{0.0228\overline{\backslash 15^\circ\, 16'}}$$

$$= 43.86\underline{/15^\circ\, 16'} \text{ ohms} \qquad \textit{Ans.}$$

$$(e)\quad R = ZG/Y = \frac{43.86 \times 0.022}{0.0228} = 42.3 \text{ ohms} \qquad \textit{Ans.}$$

$$(f)\quad X = ZB/Y = \frac{43.86 \times 0.006}{0.0228} = 11.52 \text{ ohms inductive reactance} \qquad \textit{Ans.}$$

Problem 3-17: A coil having 18.6 ohms resistance and 25.4 ohms inductive reactance is connected in parallel with a circuit consisting of 12.9 ohms resistance and 42.1 ohms capacitive reactance. Determine: (*a*) the admittance; (*b*) the equivalent impedance of the parallel circuit; (*c*) the equivalent resistance; (*d*) the equivalent reactance.

Problem 3-18: An a-c parallel circuit contains three branches. Branch one consists of 8 ohms resistance, 30 ohms inductive reactance, and 40 ohms capacitive reactance connected in series. Branch two consists of 10 ohms resistance connected in series with 15 ohms capacitive reactance. Branch three consists of 12 ohms resistance connected in series with 8 ohms inductive reactance. Determine: (*a*) the admittance; (*b*) the equivalent impedance; (*c*) the equivalent resistance; (*d*) the equivalent reactance.

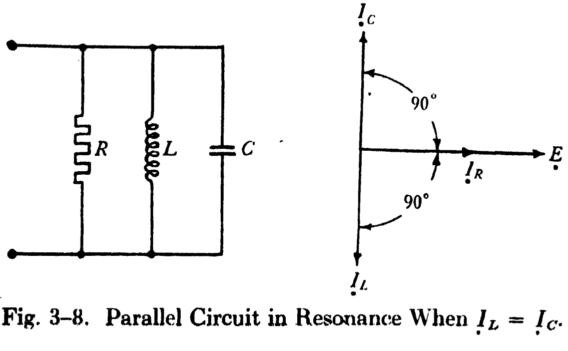


Fig. 3-8. Parallel Circuit in Resonance When $\underset{\cdot}{I}_L = \underset{\cdot}{I}_C$.

9. Resonance in A-c Parallel Circuits. In a series circuit, resonance occurs when the inductive reactance voltage X_LI equals the capacitive reactance voltage X_CI. Resonance occurs in a parallel circuit when the sum of the vertical components of the branch currents equals zero.

Assume the parallel circuit (Fig. 3-8) to contain resistance only in the first branch, inductance only in the second branch, and capacitance only in the third branch. The vector diagram is plotted with voltage along the axis of reals. I_R is in phase with E, I_L lags E by 90°, and I_C leads E by 90°. In order for resonance to occur in this type of circuit, I_L must equal I_C in magnitude, and under this condition the line current equals I_R. But before I_L can equal I_C, X_L must equal X_C. Thus, the condition for resonance in this type of parallel circuit is the same as that for a series circuit, and equation 3-7 will apply.

$$f = 1/2\pi\sqrt{LC}$$

If the resistance branch of Fig. 3-8 were removed, leaving pure inductance connected in parallel with pure capacitance, the line current at resonance would be zero, although current would flow in the branch circuits. The series equivalent impedance of this circuit becomes infinite at resonance, since Z equals E/I, and I equals zero. The wave trap (Fig. 3-9) makes use of this principle. The parallel circuit is tuned to resonate at a definite frequency by adjusting the variable condenser. Impedance to current of this frequency is practically infinite, whereas impedance to currents of other than the resonant frequency is small.

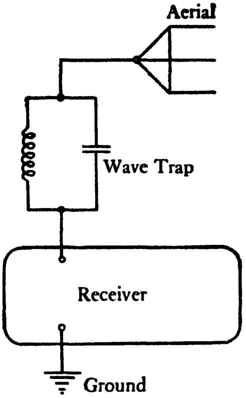


Fig. 3-9. Wave Trap.

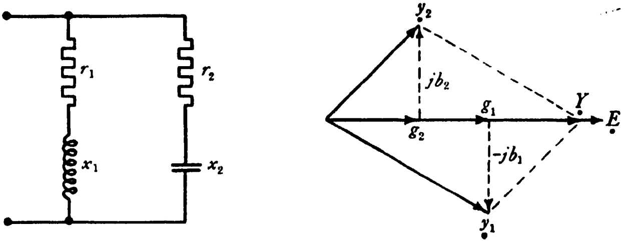


Fig. 3-10. Parallel Circuit in Resonance When $b_1 = b_2$.

If the parallel circuit (Fig. 3-10) contains resistance and inductance in series in one branch, and resistance and capacitance in series in a second branch, equation 3-27 will not hold, except when r_1 equals r_2. However, resonance will occur when the vertical component of I_1 is equal and opposite to the vertical component of I_2. Based on this assumption an equation may be derived. At resonance,

$$I_1 \sin \theta_1 = I_2 \sin \theta_2$$

But

$$I_1 \sin \theta_1 = Eb_1, \text{ and } I_2 \sin \theta_2 = Eb_2$$

Hence, at resonance,

$$Eb_1 = Eb_2$$

and

$$b_1 = b_2$$

Substituting for b_1 and b_2 their values, we have

$$\frac{x_1}{r_1^2 + x_1^2} = \frac{x_2}{r_2^2 + x_2^2}$$

Substituting for x_1 its value ωL, and for x_2 its value $\frac{1}{\omega C}$,

$$\frac{\omega L}{r_1^2 + (\omega L)^2} = \frac{\frac{1}{\omega C}}{r_2^2 + \frac{1}{(\omega C)^2}}$$

By cross multiplication,

$$\omega L r_2^2 + \frac{\omega L}{(\omega C)^2} = \frac{r_1^2}{\omega C} + \frac{(\omega L)^2}{\omega C}$$

$$\omega^3 r_2^2 LC^2 + \omega L = \omega r_1^2 C + \omega^3 L^2 C$$

Dividing by ω and transposing,

$$\omega^2 r_2^2 LC^2 - \omega^2 L^2 C = r_1^2 C - L$$

$$\omega^2 = \frac{r_1^2 C - L}{r_2^2 LC^2 - L^2 C}$$

$$\omega = 2\pi f = \sqrt{\frac{r_1^2 C - L}{r_2^2 LC^2 - L^2 C}}$$

$$f = \frac{1}{2\pi\sqrt{LC}} \cdot \sqrt{\frac{r_1^2 C - L}{r_2^2 C - L}} \qquad (3\text{-}42)$$

In cases where $r_1^2 C > L$ and $r_2^2 C < L$, the quantity $\left[\frac{r_1^2 C - L}{r_2^2 C - L}\right]$ becomes negative, and its square root is imaginary. This quantity is likewise imaginary when $r_1^2 C < L$ and $r_2^2\, C > L$. Under these conditions there is no value of frequency at which resonance will occur. When r_1 equals r_2 the quantity $\sqrt{\frac{r_1^2 C - L}{r_2^2 C - L}}$ becomes unity and $f = 1/2\pi\sqrt{LC}$, which is the same equation as that for resonance frequency of a series circuit.

10. Parallel Circuit Resonant at All Frequencies. A parallel circuit of two branches, similar to that shown in Fig. 3-10, will resonate at all frequencies when

$$r_1 = r_2 = \sqrt{L/C} \qquad (3\text{-}43)$$

This can be proved as follows: Resonance occurs when b_1 equals b_2. That is, when

$$\frac{\omega L}{r_1^2 + (\omega L)^2} = \frac{\frac{1}{\omega C}}{r_2^2 + \frac{1}{(\omega C)^2}}$$

Simplifying the left-hand member of the above equation by dividing numerator and denominator by ωL, and simplifying the right-hand member by multiplying both numerator and denominator by ωC, we have

$$\frac{1}{\frac{r_1^2}{\omega L} + \omega L} = \frac{1}{r_2^2\omega C + \frac{1}{\omega C}}$$

Multiplying both denominators by ω,

$$\frac{1}{\frac{r_1^2}{L} + \omega^2 L} \qquad \frac{1}{r_2^2\omega^2 C + \frac{1}{C}}$$

Now when

$$r_1^2/L = 1/C \text{ or } r_1 = \sqrt{L/C}$$

and

$$r_2^2 C = L \text{ or } r_2 = \sqrt{L/C}$$

the equation would become

$$\frac{1}{\frac{1}{C} + \omega^2 L} = \frac{1}{\omega^2 L + \frac{1}{C}}$$

The two sides of the equation are identical and would be equal at all frequencies. Hence, resonance will occur for this particular case at all frequencies, and the circuit behaves like a pure resistance circuit.

Problem 3-19: A parallel circuit with three branches contains a pure inductance of 0.25 henry in branch one, 35-mfd capacitance in branch two, and 300 ohms noninductive resistance in branch three. An a-c voltage held constant in magnitude at 110 volts, but with frequencies varying from 30 to 80 cycles per second is applied to the circuit. (*a*) Determine resonant frequency for the circuit; (*b*) construct a data sheet with columns for: f, I_L, I_C, I_R, $I/\underline{\theta}$, and $Z/\underline{\theta}$. Fill in data at 30, 40, 50, 60, 70, and 80 cycles per second, as well as at resonant frequency. (*c*) With frequency as abscissa, plot the magnitudes of the following as ordinate: I_L, I_C, I_R, I, and Z. Assume resistance constant at all frequencies.

Problem 3-20: Derive the equation for resonant frequency of a parallel circuit having two branches; branch one contains resistance and inductance in series, and branch two contains pure capacitance.

Problem 3-21: At what frequency will resonance occur in a parallel circuit having two branches as follows: branch one consists of 15 ohms resistance in series with 0.006-henry inductance; and branch two consists of 8 ohms resistance in series with 40-mfd capacitance?

Problem 3-22: Branch one of a parallel circuit contains 20 ohms resistance in series with 0.002-henry inductance. Branch two contains 20 ohms resistance in series with 5-mfd capacitance. At what frequency will the parallel circuit resonate?

11. Power in Single-Phase Circuits. When the voltage and current waves in a single-phase circuit are true sine waves, the power taken by the circuit equals the product of the volt-amperes and the cosine of the phase angle between the voltage and current waves. (See Chapter I, Article 24.) That is,

$$P = EI \cos \theta$$

where $\cos \theta$ is the power factor of the circuit.

This equation has been used previously in this text only where either $\underset{\cdot}{E}$ or $\underset{\cdot}{I}$ was considered to lie along the axis of reals. It can be used, however, regardless of the positions of $\underset{\cdot}{E}$ and $\dot{I}$ on the vector diagram. For example, consider the vectors $E\underline{/\alpha}$ and $I\underline{/\beta}$ (Fig. 3-11) to represent the voltage and current of a single-phase circuit. The power taken by the circuit would be:

$$P = EI \cos (\alpha - \beta)$$
$$\theta = (\alpha - \beta)$$
$$P = EI \cos \theta$$

Attention is called to the fact that *power is not the product of* $E\underline{/\alpha}$ *and* $I\underline{/\beta}$, *but the product of their magnitudes and the cosine of the difference of their direction angles.*

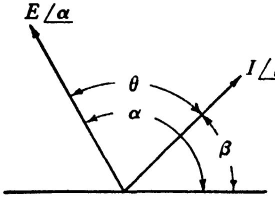


Fig. 3-11. Power = $EI \cos \theta$.

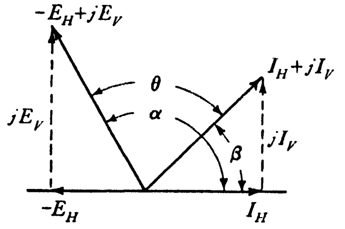


Fig. 3-12. Power $= - E_H I_H + E_V I_V$.

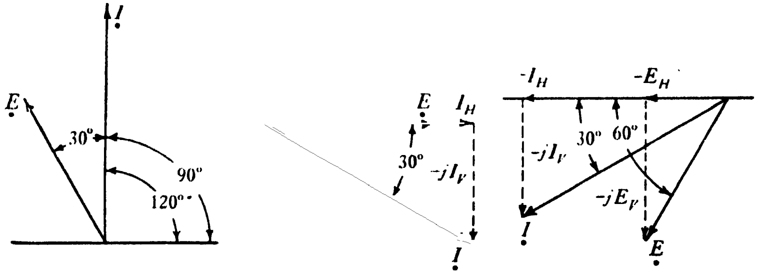
When the vectors of voltage and current are expressed in rectangular coordinates, power equals the product of the horizontal components added algebraically to the product of the vertical components. For example, consider the vector diagram of the single-phase circuit, Fig. 3-12.

$$P = EI \cos (\alpha - \beta)$$

Expanding $\cos (\alpha - \beta)$ and simplifying:

$$P = EI (\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$
$$P = EI \cos \alpha \cos \beta + EI \sin \alpha \sin \beta$$
$$P = E \cos \alpha I \cos \beta + E \sin \alpha I \sin \beta$$
$$P = E_H I_H + E_V I_V \text{ watts} \qquad (3\text{-}44)$$

The positive and negative sign preceding the horizontal and vertical components must be considered when determining power by use of equation 3-44, but *the complex operator j must not be included in computing the product of the vertical components.*



(a) (b) (c)

Fig. 3–13. Vector Diagrams for Example 3–9.

Example 3-9: A current of 10 amp flows when a single-phase circuit is placed across a 440-volt a-c source. This current lags the voltage by 30°. Determine the power drawn by the circuit: (*a*) with vectors expressed in polar coordinates, and $\underset{\cdot}{E}$ plotted 120° ahead of the reference axis; (*b*) with $\underset{\cdot}{E}$ and $\underset{\cdot}{I}$ expressed in rectangular coordinates, and $\underset{\cdot}{E}$ plotted along the axis of reals; (*c*) with $\underset{\cdot}{E}$ and $\underset{\cdot}{I}$ expressed in rectangular coordinates, and $\underset{\cdot}{E}$ placed 240° ahead of the axis of reals.

Solution. See Fig. 3-13.

(*a*) $P = EI\cos(120° - 90°) = EI\cos 30°$
$= 440 \times 10 \times 0.866 = 3810$ watts *Ans.*

(*b*) $\underset{\cdot}{E} = 440 + j\,0$
$\underset{\cdot}{I} = 8.66 - j\,5$
$\underset{\cdot}{P} = (440 \times 8.66) + (0 \times [-5]) = 3810$ watts. *Ans.*

(*c*) $\underset{\cdot}{E} = -220 - j\,381$
$\underset{\cdot}{I} = -8.66 - j\,5$
$\underset{\cdot}{P} = (-220 \times -8.66) + (-381 \times -5)$
$= 1905 + 1905 = 3810$ watts. *Ans.*

When the total energy dissipated by a single-phase circuit is in the form of heat, the power taken by the circuit equals the product of the effective resistance and the square of the current. The proof of this statement is as follows:

$$P = EI\cos\theta$$
$$E\cos\theta = RI$$
$$P = RII = RI^2$$

The total power drawn by a parallel circuit equals the arithmetic sum of the power drawn by the branch circuits. This relationship is illustrated by example 3-10.

Example 3-10: A parallel circuit consisting of two branches is connected across a 550-volt 60-cycle source. Branch one has 4 ohms resistance connected in series with 8 ohms inductive reactance, and branch two has 5 ohms resistance connected in series with 12 ohms capacitive reactance. Determine: (*a*) the power drawn by each branch; (*b*) the power drawn by the circuit; (*c*) the power factor of the parallel circuit.

Solution. Assuming voltage along the reference axis.

$$(a)\quad \underset{\cdot}{I}_1 = \frac{E + j\,0}{r_1 + j\,x_1} = \frac{550}{4 + j\,8}$$

$$= \frac{550}{4 + j\,8} \cdot \frac{4 - j\,8}{4 - j\,8} = \frac{2200 - j\,4400}{80}$$

$$= 27.5 - j\,55 \text{ amp}$$

$$P_1 = (550 \times 27.5) + (0 \times [-55]) = 15{,}125 \text{ watts} \qquad \textit{Ans.}$$

$$\underset{\cdot}{I}_2 = \frac{E + j\,0}{r_2 - j\,x_2} = \frac{550}{5 - j\,12}$$

$$= \frac{550}{5 - j\,12} \cdot \frac{5 + j\,12}{5 + j\,12} = \frac{2750 + j\,6600}{169}$$

$$= 16.3 + j\,39 \text{ amp}$$

$$P_2 = (550 \times 16.3) + (0 \times 39) = 8{,}965 \text{ watts} \qquad \textit{Ans.}$$

$$(b)\quad P = P_1 + P_2 = 15{,}125 + 8{,}965 = 24{,}090 \text{ watts} \qquad \textit{Ans.}$$

$$(c)\quad \underset{\cdot}{I} = \underset{\cdot}{I}_1 + \underset{\cdot}{I}_2 = 27.5 - j\,55 + 16.3 + j\,39$$

$$= 43.8 - j\,16 \text{ amp}$$

$$I\underline{/\theta} = \sqrt{43.8^2 + (-16)^2}\,\underline{/\tan^{-1} - 16/43.8}$$

$$= 46.6\overline{\backslash 20^\circ\, 6'} \text{ amp}$$

$$\text{Power factor} = \cos 20^\circ\, 6' = 0.939 \qquad \textit{Ans.}$$

Check on Part (*b*)

$$P = EI\cos\theta = 550 \times 46.6 \times 0.939 = 24{,}090 \text{ watts} \qquad \textit{Ans.}$$

Alternate Solution for Part (*a*): With $\underset{\cdot}{E}$ along the reference axis, determine $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$ as in the previous solution. Expressing $\underset{\cdot}{I}_1$ and $\underset{\cdot}{I}_2$ in polar coordinates, we have:

$$I_1\underline{/\theta_1} = \sqrt{(27.5)^2 + (-55)^2}\,\underline{/\tan^{-1} - 55/27.5}$$

$$= 61.5\overline{\backslash 63^\circ\, 26'} \text{ amp}$$

$$P_1 = EI_1 \cos 63^\circ\, 26' = 550 \times 61.5 \times .447$$

$$= 15{,}125 \text{ watts} \qquad \textit{Ans.}$$

Also,

$$P_1 = R_1 I_1^2 = 4 \times 61.5^2 = 15{,}125 \text{ watts} \qquad \textit{Ans.}$$

$$I_2\underline{/\theta_2} = \sqrt{16.3^2 + 39^2}\ \underline{/\tan^{-1} 39/16.3}$$
$$= 42.26\underline{/67^\circ\ 18'} \text{ amp}$$
$$P_2 = EI_2 \cos 67^\circ\ 18' = 550 \times 42.26 \times 0.386$$
$$= 8{,}965 \text{ watts} \qquad \textit{Ans.}$$

Also,

$$P_2 = R_2 I_2^2 = 5 \times 42.26^2 = 8{,}965 \text{ watts} \qquad \textit{Ans.}$$

Problem 3-23: An a-c voltage $\underset{\cdot}{E} = -110 + j\,191$ volts is applied to a circuit and causes a current $\underset{\cdot}{I} = -2 + j\,11$ amp. Determine the power drawn by the circuit.

Problem 3-24: A current $I = 4.2\overline{\backslash 12^\circ\ 42^3}$ amp flows in a circuit when $\underset{\cdot}{E} = 115\underline{/17^\circ\ 12'}$ volts is applied. Determine: (*a*) the power drawn by the circuit; (*b*) the power factor of the circuit.

Problem 3-25: A single-phase circuit has an effective resistance of 3.92 ohms and an inductive reactance of 7.84 ohms at 60 cps. The current in the circuit is 11.4 amp. What power is drawn by the circuit?

Problem 3-26: An inductance of 0.325 henry is connected in series with a non-inductive resistance of 50 ohms across a 220-volt, 25-cycle source. (*a*) What power is taken by the circuit? (*b*) What is the power factor of the circuit?

Problem 3-27: (*a*) Determine the power drawn by the circuit of problem 3-26 if the voltage applied is changed to 220 volts, 60 cycles. (*b*) Also, determine the power factor.

Problem 3-28: A parallel circuit has two branches. Branch one contains 30 ohms resistance in series with 40 ohms inductive reactance at 60 cps, and branch two contains 60 ohms resistance in series with 30 ohms inductive reactance and 60 ohms capacitive reactance at 60 cps. The parallel circuit is connected across a 240-volt, 60-cycle source. Determine: (*a*) the power required by each branch circuit; (*b*) the power required by the parallel circuit; (*c*) the power factor of the parallel circuit.

12. To Determine R, X_L, and L from Measurements.

A single-phase circuit having resistance and inductance is connected across an a-c source, as shown in Fig. 3-14. An ammeter, a voltmeter, and a wattmeter are connected in the circuit. The wattage divided by the volt-amperes is the power factor ($\cos\theta$). The phase angle θ may then be obtained from tables. The magnitudes of E and I are indicated on the instruments, and E divided by $I\overline{\backslash\theta}$ will give $Z\underline{/\theta}$. $Z\cos\theta$ equals R, and $Z\sin\theta$ equals X_L. If the frequency of the source is known, L in henries may be computed by dividing X_L by $2\pi f$.

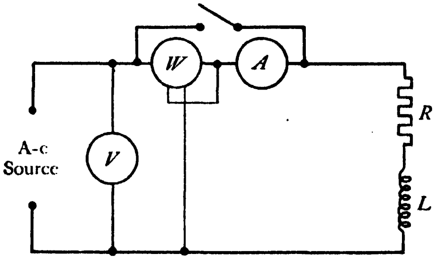


Fig. 3-14. Circuit for Determining R, X_L, and L.

An alternate method would be to determine the magnitude of Z, which equals the magnitude of E divided by that of I. R equals P/I^2, and X_L equals $\sqrt{Z^2 - R^2}$. L equals $X_L \div 2\pi f$.

Example 3-11: A single-phase circuit is placed across a 120-volt, 60-cycle source, with an ammeter, a voltmeter, and a wattmeter connected, as shown in Fig. 3-14. The instruments indicate 12 amp, 120 volts, and 800 watts. Determine: (*a*) the power factor; (*b*) the phase angle; (*c*) the impedance; (*d*) the resistance; (*e*) the inductive reactance; (*f*) the inductance.

Solution.

(*a*) $\cos\theta = P/EI = 800/120 \times 12 = 0.555$ *Ans.*

(*b*) $\theta = \cos^{-1} 0.555 = 56° 17'$ *Ans.*

(*c*) $Z\underline{/\theta} = \dfrac{E}{I\backslash\theta} = \dfrac{120}{12\backslash 56° 17'} = \underline{10\underline{/56° 17'}\text{ ohms}}$ *Ans.*

(*d*) $R = Z\cos\theta = 10 \times 0.555 = 5.55$ ohms *Ans.*

(*e*) $X_L = Z\sin\theta = 10 \times 0.832 = 8.32$ ohms *Ans.*

(*f*) $L = X_L/2\pi f = 8.32/6.28 \times 60 = 0.0221$ henry *Ans.*

Alternate Solution for Parts (*d*) and (*e*)

(*d*) $R = P/I_2 = 800/12^2 = 5.55$ ohms *Ans.*

(*e*) In magnitude $Z = E/I = 120/12 = 10$ ohms

$X_L = \sqrt{Z^2 - R^2} = \sqrt{10^2 - 5.55^2} = 8.32$ ohms *Ans.*

Problem 3-29: A coil is placed in series with a resistance of 20 ohms across a 220-volt, 50-cycle source, and draws 4.2 amp and 670 watts. Determine: (*a*) the power factor and phase angle of the circuit; (*b*) the circuit impedance; (*c*) the circuit resistance; (*d*) the coil resistance; (*e*) the coil reactance; (*f*) the inductance of the coil; (*g*) the power factor and phase angle of the coil.

Problem 3-30: An air-core coil is connected across a 240-volt, 60-cycle source, and draws 10 amp. If the effective resistance of the coil is 5 ohms, determine: (*a*) the impedance; (*b*) the reactance; (*c*) the power drawn by the coil.

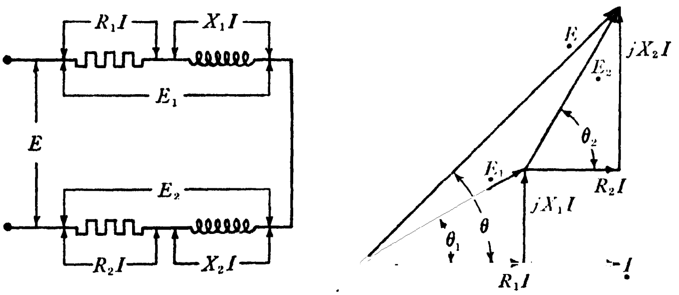


Fig. 3-15. Diagram for Single-Phase Transmission with an Inductive Load.

13. Voltage Drop in A-c Series Circuits.

In dealing with a-c transmission and distribution systems the effect of capacitance is considered negligible except in long high-voltage lines, and in cables operating at relatively high voltages. Resistance and inductive reactance of the line, however, must be considered in determining the voltage drop between the source and the load. Fig. 3-15 shows the diagrams for an inductive load at the end of a single-phase transmission line. The resistive and reactive voltages for both wires of the line are indicated schematically as though they were concentrated at one position in the line. Since the load voltage E_1 and the line impedance voltage E_2 are often out of phase, they must be added vectorially to obtain the voltage of the source E. Expressed in rectangular coordinates,

$$\underset{\cdot}{E} = (R_1I + R_2I) + j\,(X_1I + X_2I)$$

Substituting for $\underset{\cdot}{E}$ its value $\underset{\cdot}{Z}I$ and dividing by I, we have

$$\underset{\cdot}{Z} = (R_1 + R_2) + j\,(X_1 + X_2)$$

A capacitive load is shown at the end of a single-phase transmission line in Fig. 3-16. The load voltage $\underset{\cdot}{E}_1$ is plotted θ_1° behind the current, where θ_1 is the phase angle of the load. The line resistance voltage R_2I which is in phase with the current and the line reactance voltage X_2I which is 90° ahead of the current are added to $\underset{\cdot}{E}_1$ to give the voltage of the source $\underset{\cdot}{E}$. It is evident that the voltage regulation of single-phase transmission and distribution systems is materially affected, not only by the resistance and reactance of the line, but by the power factor of the load as well.

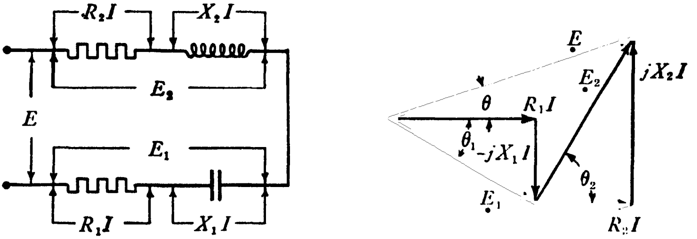


Fig. 3-16. Diagrams for a Single-Phase Transmission Line with a Capacitive Load.

Example 3-12: A 20-kw, 440-volt, 60-cycle, 80 per cent leading power factor load is connected at one end of an overhead single-phase distribution line. The line is 800 feet long and consists of two No. 6 A. W. G. solid copper conductors spaced 24 inches apart. Assume the resistance of each single conductor per mile to be 2.11 ohms and the inductive reactance of each single conductor per mile at 60 cycles with 24 inch spacing to be 0.72 ohms. Consider the capacitance of the line to be negligible. Determine the magnitude and phase angle of the voltage at the input end of the line when the load voltage is 440 volts, 60 cycles.

Solution. (See Fig. 3-16.) The magnitude of current is,

$$I = P/E\cos\theta = 20{,}000/440 \times 0.8 = 56.9 \text{ amp}$$

With current plotted along the axis of reals the load voltage $\underset{\bullet}{E}_1$ expressed in rectangular coordinates is

$$E_1\overline{\backslash\theta_1} = E_1 (\cos\theta_1 - j\sin\theta_1) = 352 - j\,264 \text{ volts}$$

The resistance of the line,

$$R_2 = \frac{2.11 \times 800 \times 2}{5280} = 0.64 \text{ ohm}$$

The line resistance voltage,

$$R_2I = 0.64 \times 56.9 = 36.6 \text{ volts}$$

The inductive reactance of the line,

$$X_2 = \frac{0.72 \times 800 \times 2}{5280} = 0.218 \text{ ohm}$$

The line inductive reactance voltage,

$$X_2I = 0.218 \times 56.9 = 12.4 \text{ volts}$$

The line impedance voltage in rectangular coordinates,

$$\underset{\bullet}{Z}_2I = 36.6 + j\,12.4 \text{ volts}$$

The input voltage in rectangular coordinates,

$$\begin{aligned}\underset{\bullet}{Z}I &= 352 - j\,264 + 36.6 + j\,12.4\\ &= 388.6 - j\,251.6 \text{ volts}\end{aligned}$$

The input voltage in polar coordinates,

$$\begin{aligned}ZI\underline{/\theta} &= \sqrt{(388.6)^2 + (-251.6)^2}\,\underline{/\tan^{-1} - 251.6/388.6}\\ &= 463.5\overline{\backslash 32^\circ 55'} \text{ volts} \qquad \textit{Ans.}\end{aligned}$$

In Fig. 3-17 an impedance having resistance and inductive reactance is connected in series with a single-phase induction motor in order to operate

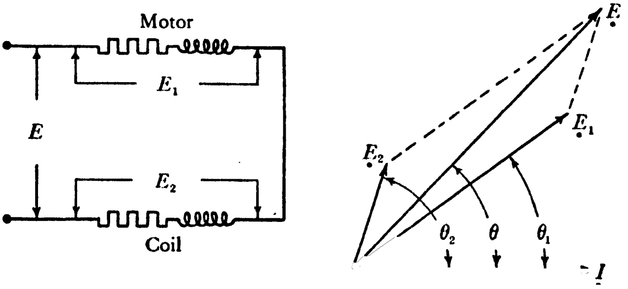


Fig. 3-17. Diagrams for Example 3-13.

the motor at less than line voltage. The magnitude of the line voltage, the phase angle of the series impedance voltage referred to the current, and both magnitude and phase angle of the motor voltage are known. The phase angle of the line voltage referred to the current and the magnitude of the series impedance voltage are unknown. One method of solving this type of problem is illustrated in example 3-13.

Example 3-13: A single-phase induction motor draws 760 watts at 80 per cent lagging power factor, when 110 volts, 60 cycles, are applied to its terminals. The motor is to operate from a 150-volt, 60-cycle source with the line voltage reduced to 110 volts across the motor by placing a coil having 30 per cent lagging power factor in series. Determine the resistance and reactance of the coil.

Solution. See Fig. 3-17.

(*a*) Let $\underset{\cdot}{E_1}$ represent the motor voltage, $\underset{\cdot}{E_2}$ the coil voltage, and $\underset{\cdot}{E}$ the line voltage. Likewise, let θ_1 represent the phase angle of the motor, θ_2 that of the coil, and θ the phase angle of the circuit. Then

$$\underset{\cdot}{E} = (E_1 \cos\theta_1 + E_2 \cos\theta_2) + j\,(E_1 \sin\theta_1 + E_2 \sin\theta_2)$$

The magnitude of $\underset{\cdot}{E}$

$$= \sqrt{(E_1 \cos\theta_1 + E_2 \cos\theta_2)^2 + (E_1 \sin\theta_1 + E_2 \sin\theta_2)^2}$$

$$= \sqrt{E_1^2 \cos\theta_1^2 + 2E_1E_2 \cos\theta_1 \cos\theta_2 + E_2^2 \cos\theta_2^2 + E_1^2 \sin\theta_1^2 + 2E_1E_2 \sin\theta_1 \sin\theta_2 + E_2^2 \sin\theta_2^2}$$

$$= \sqrt{E_1^2 (\sin\theta_1^2 + \cos\theta_1^2) + E_2^2 (\sin\theta_2^2 + \cos\theta_2^2) + 2\,E_1E_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)}$$

But

$$(\sin\theta_1^2 + \cos\theta_1^2) = 1, \text{ and } (\sin\theta_2^2 + \cos\theta_2^2) = 1$$

Also

$$(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) = \cos(\theta_2 - \theta_1)$$

Substituting these identities, in magnitude

$$E = \sqrt{E_1^2 + E_2^2 + 2E_1E_2 \cos(\theta_2 - \theta_1)}$$

(The above equation is an application of the law of cosines.)

Substituting values from example 3-13 and simplifying by squaring both sides, we have

$$150^2 = 110^2 + E_2^2 + 2 \times 110 \times E_2 \cos(72°\,33' - 36°\,52')$$

$$E_2^2 + (178.7)E_2 = 22{,}500 - 12{,}100$$

Completing the square of the left side of equation by adding $(178.7/2)^2$ or its equivalent value $(89.35)^2$ to each side,

$$E_2^2 + 178.7E_2 + 89.35^2 = 22{,}500 - 12{,}100 + 89.35^2$$

$$E_2 + 89.35 = \pm\,135.55$$

$$E_2 = 46.15 \text{ volts}$$

$$R_2I = E_2 \cos\theta_2 = 46.15 \times 0.3 = 13.85 \text{ volts}$$

$$X_2I = E_2 \sin\theta_2 = 46.15 \times 0.954 = 44.03 \text{ volts}$$

The magnitude of I is

$$I = \frac{P_1}{E_1 \cos \theta} = 760/110 \times 0.8 = 8.65 \text{ amp}$$

$$R_{coil} = \frac{R_2 I}{I} = 13.85/8.65 = 1.6 \text{ ohms} \qquad \textit{Ans.}$$

$$X_{coil} = \frac{X_2 I}{I} = 44.03/8.65 = 5.1 \text{ ohms} \qquad \textit{Ans.}$$

Problem 3-31: A 60-cycle, single-phase transmission line supplies a 110-kw load at 0.6 power factor. The voltage at the terminals of the load is 11,000 volts. The line is 25 miles long (50 miles of wire), and consists of two No. 4 A. W. G. solid copper conductors spaced 60 inches apart. Assume the resistance per conductor per mile for No. 4 to be 1.33 ohms, and the inductive reactance per conductor per mile at 60 cycles and for a spacing of 60 inches to be 0.8 ohms. Neglect the effect of capacitance. Determine: (*a*) the voltage applied to the transmission line; (*b*) the power factor of the line and the load taken collectively; (*c*) the efficiency of transmission.

Problem 3-32: Determine the value of noninductive resistance which might be substituted for the coil of example 3-13 and still give 110 volts across the motor terminals.

14. Power-Factor Correction. The extensive use of induction motors, magnetic controllers, certain types of electric furnaces and welding machines, as well as other inductive equipment causes the typical electric load to have a lagging power factor. As the power factor decreases, the current and kva for a definite wattage constant-voltage load increases. This is evident since $I = P/E \cos \theta$. Thus, transformers, alternators, switching equipment, and line conductors for supplying low power-factor loads must be larger, contain more copper, and cost more per kw than equipment which is constructed to supply unity power-factor loads.

A distribution or transmission line may be operating above rated current due to the low power factor of the load which it supplies. By improving the power factor of the load the line current will be reduced, the I^2R copper losses will be reduced, and the efficiency of transmission will be raised. Furthermore, the undesirable effect of lagging power factor in increasing the voltage regulation of transmission and distribution lines which was illustrated in the previous article will be diminished.

Because of these undesirable effects of low power factor, power companies are interested in encouraging customers to maintain the best possible load power factor. Contracts for large power consumers usually call for rebates when their load power factor is above 85 per cent and penalties when their power factor falls below 85 per cent. The amount of the rebate or penalty increases as the power factor varies from 85 per cent.

One method of improving lagging power factor is to connect a static condenser in parallel with the load. On polyphase systems, synchronous condensers are frequently used for this purpose. Synchronous condensers are specially designed synchronous motors which operate without mechanical load at low leading power factors. Other types of synchronous motors which drive mechanical loads are often designed to provide some power-factor correction.

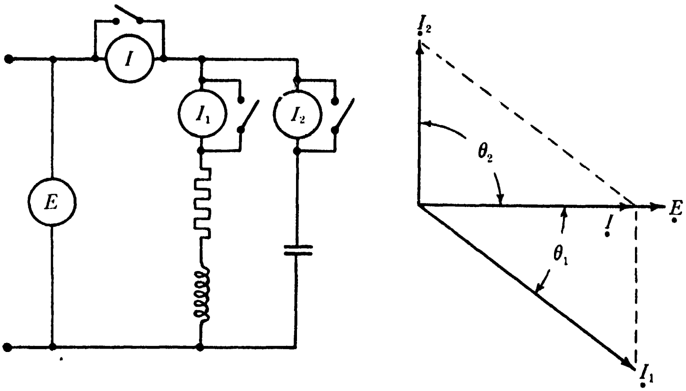


Fig. 3-18. Diagrams for Example 3-14.

Example 3-14: A single-phase motor draws 2000 watts at 80 per cent lagging power factor when connected across a 220-volt, 60-cycle source. A static condenser is connected in parallel with the load to give unity power factor. Determine the required capacitance.

Solution. See Fig. 3-18.

Let I_1 equal motor current, I_2 the charging current of condenser, and I the line current. In magnitude,

$$\begin{aligned} I_1 &= P/E \cos\theta_1 = 200/220 \times 0.8 = 11.35 \text{ amp} \\ I_2 &= I_1 \sin\theta_1 = 6.81 \text{ amp} \\ X_C &= E/I_2 = 220/6.81 = 32.3 \text{ ohms} \\ C &= 10^6/2\pi f X_C = 10^6/6.28 \times 60 \times 32.3 = 82.2 \text{ mfd} \qquad \textit{Ans.} \end{aligned}$$

Problem 3-33: The rated current for the secondary coil of a transformer is 312.5 amp, and the rated voltage of the coil is 480 volts. Determine: (*a*) its kva rating; (*b*) its kw rating for an 80 per cent power-factor load; (*c*) its kw rating for a 50 per cent power-factor load.

Problem 3-34: A single-phase induction motor draws 800 watts at 60 per cent lagging power factor when connected across a 110-volt, 60-cycle source. A static condenser is to be connected in parallel with the motor to correct the power factor to 85 per cent lagging. Determine the required capacitance.

15. Kilovolt-Ampere Diagrams. When the information concerning a-c parallel circuits is expressed in terms of kva and power factor, the solutions of certain types of problems may be simplified by use of kva diagrams.

In order to construct this type of diagram, kva must be considered as a vector quantity displaced from the axis of reals by an angle whose cosine is the power factor. The horizontal component of the kva is the product of kva and cos θ or the kw, and the vertical component is the product of the kva and sin θ or the kvar (*kilovolt-amperes reactive*). Expressing the kva in rectangular coordinates, we have,

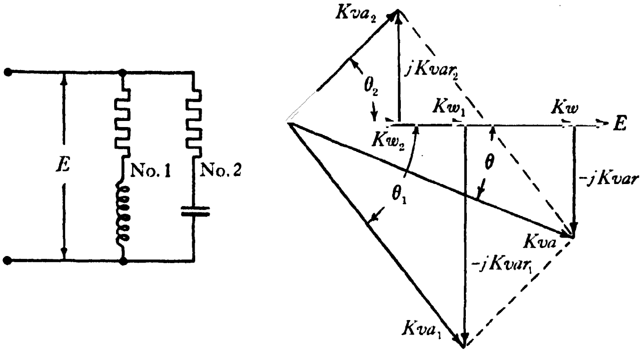


Fig. 3-19. Addition of Two Kilovolt-Ampere Loads.

$$\text{kva} = \text{kw} + j\,\text{kvar} \tag{3-45}$$

Equation 3-45 is to be read as follows: *the kilovolt-amperes equals the vector sum of the kilowatts and the kilovars.*

Fig. 3-19 shows kva$_1$ plotted θ_1 degrees in a negative direction from the axis of reals since the power factor of branch one is lagging, and kva$_2$ is plotted θ_2 degrees in a positive direction from the axis of reals since branch two has a leading power factor. The total kva of the parallel circuit is the vector sum of kva$_1$ and kva$_2$, and the power factor of the parallel circuit is the total kw divided by the total kva.

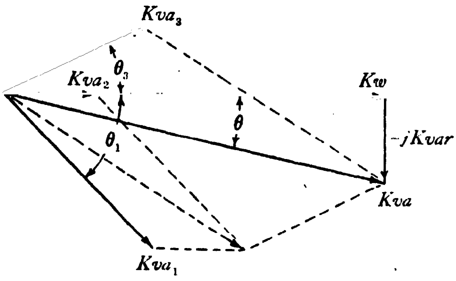


Fig. 3-20. Diagram for Example 3-15.

Example 3-15: Three single-phase distribution feeders are connected to a bus. Number one supplies a 70-kva, 70 per cent lagging power-factor load; number two supplies a 30-kva, unity power-factor load; and number three supplies a 50-kva, 90 per cent leading power-factor load. Determine: (*a*) the kva supplied by the bus; (*b*) the power factor of the combined load; and (*c*) the power taken from the bus.

Solution. See Fig. 3-20.

$$\overline{\backslash\theta_1} = \overline{\backslash\cos^{-1} 0.7} = \overline{\backslash 45^\circ\,34'}$$
$$\underline{/\theta_2} = \underline{/\cos^{-1} 1.0} = \underline{/0^\circ}$$
$$\underline{/\theta_3} = \underline{/\cos^{-1} 0.9} = \underline{/25^\circ\,10'}$$

(*a*)
$$\begin{aligned}
\text{kva}_1\overline{\backslash\theta_1} &= 49 - j\,49.99 \\
\text{kva}_2\underline{/\theta_2} &= 30 + j\,0.00 \\
\text{kva}_3\underline{/\theta_3} &= 45 + j\,21.80 \\
\hline
\text{kva}\underline{/\theta} &= 124 - j\,28.19 \\
&= \sqrt{124^2 + (-28.19)^2}\ \underline{/\tan^{-1} - 28.19/124} \\
&= 127.2\overline{\backslash 12^\circ\,48'}\ \text{kva}
\end{aligned}$$
Ans.

(*b*) Power factor $= \cos 12^\circ\,48' = 0.975$
$= 97.5$ per cent lagging *Ans.*

(*c*) Kw $= 124$ kilowatts *Ans.*

Problem 3-35: A single-phase feeder supplies the following loads: a motor circuit which draws 15 kva at 60 per cent lagging power factor; an 8-kva lighting load at unity power factor; and a 2-kva static condenser load at zero leading power factor. Determine: (*a*) the total kva; (*b*) the power factor of the feeder; (*c*) the power supplied by the feeder.

Problem 3-36: A 2300-volt, 60-cycle, single-phase bus has four branch circuits. Circuit *A* supplies a 100-kva load at 90 per cent lagging power factor; circuit *B*, 60 kva at 75 per cent lagging power factor; circuit *C*, 40 kva at unity power factor; and circuit *D*, 30 kva at zero per cent leading power factor. Determine: (*a*) the total kva supplied by the bus; (*b*) the total power; (*c*) the bus power factor.

16. Series-Parallel Circuits. To determine the impedance of a series-parallel circuit, first, find the series equivalent impedance of the parallel portion of the circuit, and add this vectorially to the impedance in series with the parallel part of the circuit.

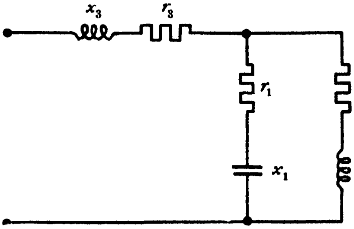


Fig. 3-21. Circuit for Example 3-16.

Example 3-16: A coil with 10 ohms resistance and 12 ohms inductive reactance is connected in series with a parallel circuit of two branches. Branch one contains 20 ohms resistance and 40 ohms capacitive reactance, and branch two 15 ohms resistance and 20 ohms inductive reactance. Determine the impedance of the series-parallel circuit.

Solution. See Fig. 3-21. Considering the parallel part of the circuit:

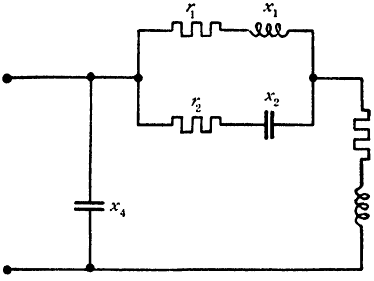
$$\begin{aligned}
y_1 &= g_1 + j\,b_1 = \frac{20}{20^2 + 40^2} + j\,\frac{40}{20^2 + 40^2} \\
&= 0.01 + j\,0.02
\end{aligned}$$

$$y_2 = g_2 - j\,b_2 = \frac{15}{15^2 + 20^2} - j\,\frac{20}{15^2 + 20^2}$$
$$= 0.024 - j\,0.032$$
$$Y = 0.01 + j\,0.02 + 0.024 - j\,0.032 = 0.034 - j\,0.012$$
$$Z_p = \frac{1}{0.034 - j\,0.012} \cdot \frac{0.034 + j\,0.012}{0.034 + j\,0.012}$$
$$= \frac{0.034 + j\,0.012}{0.0013} = 26.1 + j\,9.22 \text{ ohms}$$

Adding the impedance of the series portion:

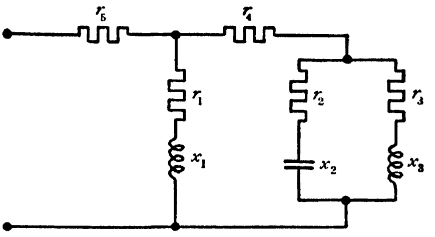
$$Z_{total} = 26.1 + j\,9.22 + 10 + j\,12 = 36.1 + j\,21.22$$
$$Z\underline{/\theta} = \sqrt{36.1^2 + 21.22^2}\,\underline{/\tan^{-1} 21.22/36.1}$$
$$= 41.9\underline{/30^\circ\,31'} \text{ ohms} \qquad \textit{Ans.}$$

Problem 3-37: Determine the impedance of the series-parallel circuit of Fig. 3-22, using the following values for resistances and reactances: $r_1 = 80$ ohms, $r_2 = 60$ ohms, $r_3 = 50$ ohms, $x_1 = 60$ ohms inductive reactance, $x_2 = 90$ ohms capacitive reactance, $x_3 = 30$ ohms inductive reactance, and $x_4 = 75$ ohms capacitive reactance.



$r_1 = 80$, $r_2 = 60$, $r_3 = 50$ ohms
$x_1 = 60$, $x_2 = 90$, $x_3 = 30$, $x_4 = 75$ ohms

Fig. 3-22. Circuit for Problem 3-37.



$r_1 = 25$, $r_2 = 30$, $r_3 = 40$, $r_4 = 45$, $r_5 = 15$ ohms
$x_1 = 50$, $x_2 = 60$, $x_3 = 35$ ohms

Fig. 3-23. Circuit for Problem 3-38.

Problem 3-38: Determine the impedance of the series-parallel circuit of Fig. 3-23, using the following values of resistances and reactances: $r_1 = 25$ ohms, $r_2 = 30$ ohms, $r_3 = 40$ ohms, $r_4 = 45$ ohms, $r_5 = 15$ ohms, $x_1 = 50$ ohms inductive reactance, $x_2 = 60$ ohms capacitive reactance, and $x_3 = 35$ ohms inductive reactance.

QUESTIONS

3-1. What is meant by the in-phase component of current? The quadrature current?

3-2. Define reactance and give its symbol. How does reactance differ from impedance?

3-3. What is meant by resonance in a series circuit? How does series resonance differ from parallel resonance?

3-4. If you were working with a series circuit and desired to produce resonance at a lower frequency, what methods could you use?

3-5. How can the tuning curve of a series circuit be made sharper without changing the value of resonant frequency?

3-6. Define: series equivalent impedance; series equivalent resistance; and series equivalent reactance.

3-7. List the steps used in determining the series equivalent impedance of a parallel circuit, when the voltage across the circuit is known.

3-8. Define admittance, conductance, and susceptance as they pertain to a-c parallel circuits. Give the symbol for each.

3-9. Under what condition will the equation for resonance in a parallel circuit of two branches be the same as that for resonance in a series circuit?

3-10. What conditions must exist in a parallel circuit having two branches to prevent resonance at any frequency?

3-11. Under what condition does resonance in a parallel circuit of two branches take place at all frequencies?

3-12. Give the equation for power in an a-c circuit when the vectors of voltage and current are expressed as $E\underline{/\alpha}$ and $I\overline{\setminus\beta}$. When $\underset{\cdot}{E} = E_H + jE_V$ and $\underset{\cdot}{I} = I_H - jI_V$.

3-13. What are the reasons why low power factor is undesirable? How may it be improved?

3-14. What is meant by kilovolt-amperes reactive?

PROBLEMS

3-39. The name plate on a single-phase induction motor reads: 2 horsepower (output), 110 volts, 60 cycles, 24 amp. Determine the power factor of the motor if the efficiency at rated output is 80 per cent.

3-40. The impedance of an air-core coil is measured at 1000 cycles as $1240\underline{/64^\circ}$ ohms. The coil is to be connected in series with a capacitor to give the circuit zero phase angle at 2000 cycles. Assuming the effective resistance as constant, determine: (*a*) the microfarads of the capacitor; (*b*) the impedance of the circuit at 2000 cycles.

3-41. A resistor and capacitor are connected in series across a 120-volt, 25-cycle source. The current has a magnitude of 6 amp when the voltage across the resistor is 75 volts. Determine the mfd capacitance of the circuit.

3-42. A resistor and capacitor are connected in series across a 440-volt, 50-cycle source. The power drawn by the circuit is 630 watts and the current magnitude is 2.6 amp. Determine: (*a*) the impedance; (*b*) the effective resistance; (*c*) the capacitance in microfarads.

3-43. An a-c single-phase motor rated at 3 horsepower (output), 220 volts, 25 cycles, is to be operated on a 260-volt, 25-cycle line. At rated load the power factor is 80 per cent lagging and the efficiency is 75 per cent. A coil having 50 per cent power factor is connected in series with the motor to reduce the voltage from 260 to 220 volts when the motor is delivering 3 horsepower. Determine the resistance and reactance of the coil.

3-44. An air-core coil is placed across a 120-volt, 60-cycle source and draws 90 watts at 40 per cent lagging power factor. The coil is then connected in series with a resistor across a 220-volt, 25-cycle source. Determine the required resistance of the series resistor if the current in the coil is to be limited to 5 amp.

3-45. Determine the inductance to be connected in series with a 0.00035-mfd capacitance to give resonance at 6×10^5 cycles per second.

3-46. A series circuit contains 25 micromicrofarads capacitance (0.000025 mfd) and 140 microhenries inductance. At what frequency is this circuit resonant?

3-47. Two coils are connected in parallel across a 110-volt, 60-cycle source. The line current has a magnitude of 12 amp and the line power factor is 80 per cent lagging. The magnitude of current in branch one is 8 amp and the power factor of this branch is 72 per cent lagging. Determine: (*a*) the current in the second coil; (*b*) the resistance and reactance of the second coil.

3-48. A coil is connected across a 220-volt, 60-cycle source, and an ammeter in the circuit indicates 6 amp. The coil is replaced by a noninductive resistance and the ammeter shows 5 amp. The coil and the resistor are connected in parallel across the source and the ammeter connected to read the line current indicates 9 amp. Determine: (*a*) the power factor of the coil; (*b*) the resistance and reactance of the coil.

3-49. A parallel circuit contains 0.0001-ohm resistance and 0.002-henry inductance in branch one, and 0.0005 microfarad in branch two. At what frequency is this circuit resonant?

3-50. A noninductive resistance and a capacitor are connected in series across a 220-volt, 60-cycle source. The resistance is adjusted so that an ammeter in the circuit indicates 8 amp and a wattmeter indicates 250 watts. Determine the resistance and the capacitance of the circuit.

3-51. A 60-cycle, single-phase transmission line 5 miles long delivers power to a 300-kw, 80 per cent lagging power-factor load, and at a load voltage of 6600 volts. The line consists of No. 0 A. W. G. solid copper conductors spaced 24 inches apart. Assuming resistance to be 0.52 ohm per mile of conductor, and inductive reactance at 60 cycles and for 24-inch spacing to be 0.636 ohm per mile of conductor, determine: (*a*) the voltage at the input end of the transmission line; (*b*) the power factor of the load and line taken collectively; (*c*) the per cent efficiency of the transmission line.

3-52. The values for the circuit of Fig. 3-24 are: $R = 200$ ohms, $L = 0.01$ henry, and $C = 1.5$ mfd. Determine the impedance of the circuit at 800 cycles per second.

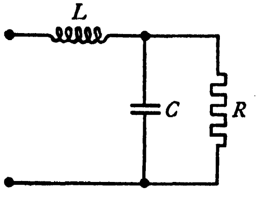


Fig. 3–24. Circuit for Problem 3–52.

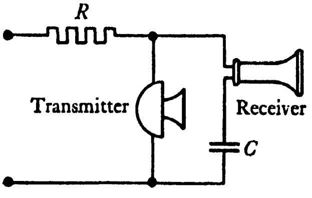


Fig. 3–25. Circuit for Problem 3–53.

3-53. The transmitter of Fig. 3-25 has an impedance of $55 + j\,0$, the receiver impedance is $150 + j\,165$, R is 45 ohms, and C in series with the receiver is 1.2 mfd. The circuit is connected across a 5-volt, 800-cycle source. Determine: (*a*) the circuit impedance; (*b*) the current in the line; (*c*) the power drawn by the transmitter; (*d*) the power drawn by the receiver.

LABORATORY EXPERIMENTS

EXPERIMENT 3-1

Purpose. To determine the effect of varying the frequency from below to above resonant frequency in an a-c series circuit containing resistance, inductance, and capacitance.

Procedure. Drive an alternator with an adjustable speed d-c motor so as to generate a voltage with frequencies ranging from about 35 to 70 cps. Connect adequate rheostat capacity in the field of the alternator so that the magnitude of voltage may be maintained constant at about 120 volts throughout the range of frequency required for the test. Connect a resistor, coil, and capacitor in series across the variable frequency source, and insert instruments to measure the impressed voltage E, current I, wattage of the entire circuit P, voltage across the resistor E_1, voltage across the coil E_2, wattage taken by the coil P_2, and voltage across the capacitor E_3. One voltmeter and one wattmeter may be used for all voltage and wattage readings.

Energize the circuit, and vary the frequency, adjusting the voltage so as to maintain constant magnitude throughout the frequency range. Check the instruments to see: (1) that the current and voltage ratings of the instruments and equipment are not exceeded; (2) that the indications of the instruments are sufficiently high on the scales to obtain reasonably accurate measurements; (3) and that resonant frequency occurs between 50 and 60 cps. If this check indicates a need for adjustment in the circuit or for a change of instruments, make the necessary alterations and repeat the check before taking data.

After the circuit is ready for the test, apply the low-frequency voltage. Read and record values of: f, E, I, P, E_1, E_2, P_2, and E_3. Repeat the readings for eight to ten values of frequency over the range suggested. One reading should be obtained with the frequency as near as possible to resonant frequency.

Report. 1. Submit a written report.

2. The data sheet should include, in addition to the values read, the following computed data for each frequency used in the test: the circuit impedance $Z\underline{/\theta}$, the circuit power factor PF, the circuit resistance R, the circuit reactance X, the coil impedance $Z_2\underline{/\theta_2}$, the inductive reactance X_L, the capacitive reactance X_C, the inductance L, and the capacitance C.

3. The report should include a curve sheet with f plotted as abscissa and the following quantities plotted as ordinates: I, Z, R, X_L, and X_C.

4. The report should include two vector diagrams: one for a frequency below resonance and the other for a frequency above resonance. The values of E_1, E_2, and E_3 should be plotted to scale and added vectorially to obtain E. The values of E from the vector diagrams should then be compared with the values of E as read.

EXPERIMENT 3-2

Purpose. To determine the effect of varying the frequency from below to above resonant frequency in an a-c parallel circuit.

Procedure. Connect the alternator to provide a voltage of constant magnitude at frequencies from about 35 to 70 cps (see suggestions, experiment 3-1). The test circuit should consist of a coil in branch one and a capacitor in branch two. Insert instruments to read the impressed voltage E, the line current I, the power P, the current in branch one I_1, and the current in branch two I_2.

Energize the circuit and vary the frequency, making the necessary field adjustments to give a constant voltage throughout the frequency range. Check the instruments to see: (1) that the current and voltage ratings of the instruments and equipment are not exceeded; (2) that the indications are high enough on the scales of the instruments to obtain reasonably accurate measurements; and (3) that resonant frequency occurs between 50 and 60 cps. If the check indicates a need for adjustments in the circuit, or for a change of instruments, make the necessary alterations and repeat the check before taking data.

After the circuit is ready for the test, apply the low frequency voltage. Read and record values of f, E, I, P, I_1, and I_2. Repeat the readings for eight or ten values of frequency over the range suggested. One reading should be obtained with frequency as near as possible to resonant frequency.

Report. **1.** Submit a written report.

2. The data sheet should include, in addition to the values read, the following computed data for each frequency used in the test: I_1 expressed in rectangular coordinates; I_2 expressed in rectangular coordinates; $I\underline{/\theta}$ computed by adding I_1 and I_2 vectorially; $Z\underline{/\theta}$ computed from E as read and $I\underline{/\theta}$ as computed; the circuit power factor PF, which is the cos of the phase angle as computed; the series equivalent resistance R; and the series equivalent reactance X.

Suggestion: Assume the phase angle of I_2 in respect to E to be 90° and the power taken by the capacitor to be zero. The entire power of the circuit will then be taken by branch one, and $\theta_1 = \cos^{-1} P/EI_1$.

The use of the computed value of line current is recommended above due to the fact that the accuracy of the measured value of current may be affected by the distortion of the wave form.

3. The report should include a curve sheet with f plotted as abscissa and the following plotted as ordinates: I as read; I as computed; $I_1 \sin \theta_1$; I_2; and PF.

CHAPTER IV

POLYPHASE SYSTEMS

1. Advantages of the Polyphase System. A multiple-phase or polyphase system is a combination of two or more single-phase systems, the voltages of which are displaced in time phase from one another. The single-phase systems which comprise the polyphase system usually are interconnected.

Polyphase systems have many advantages when compared with single-phase systems, particularly when motors above fractional horsepower sizes constitute the principle load on the system. Power in the polyphase system is relatively uniform while that of the single-phase system is pulsating. Thus, the torque produced by polyphase motors is relatively smooth when compared to the torque developed by single-phase motors. In this respect polyphase and single-phase motors are similar to multiple- and single-cylinder gasoline engines. When compared with single-phase motors, polyphase motors are simple in construction, require less maintenance, and have higher efficiencies. In general, single-phase motors with ratings above fractional horsepower sizes are not used unless other considerations outweigh the advantages of the polyphase motor.

The power rating of a given motor or generator is greater when operated as a polyphase machine than when operated single phase. For example, if the single-phase rating is taken as 100 per cent, the 2-phase rating of the same machine will be about 40 per cent greater, and the 3-phase or 6-phase rating of the same machine will be about 50 per cent greater. The amount of copper required by a 3-phase system to transmit power over a fixed distance, with equal power loss, and with equal voltages between conductors is 75 per cent of that required by a single-phase system.

A single-phase load may be energized from a polyphase system simply by utilizing one of the single-phase voltages of the system. It is not possible, however, to energize a polyphase load from a single-phase system without first converting the single-phase system into a polyphase system. The equipment for this conversion is somewhat complicated and expensive.

2. Generation of Polyphase Voltages. The stator or armature of the 2-pole alternator (Fig. 4-1) is wound with 3 coils, *aa′*, *bb′*, and *cc′*, which are electrically insulated from one another. Coil *bb′* is displaced 120° in a clockwise direction from coil *aa′*, and coil *cc′* is displaced 120° clockwise from coil *bb′* and 240° clockwise from coil *aa′*. The direction of the emf of coil *aa′* is in at *a′* and out at *a* when the south pole is above *a* and the direction of rotation is clockwise. (To verify this condition with the right-hand generator rule, the conductors must be considered to move in the opposite or counterclockwise direction in respect to the field.) Also, the potential at *a* reaches its positive maximum with clockwise rotation of the field when the center of the south pole is directly above *a*. The potential at *b* is a positive maximum at 120 time-degrees after *a* has passed through its positive maximum. At 120 time-degrees later the potential of *c* reaches its positive maximum. That is, the emfs of the three coils are displaced from one another by 120 time-degrees and form a *3-phase* system.

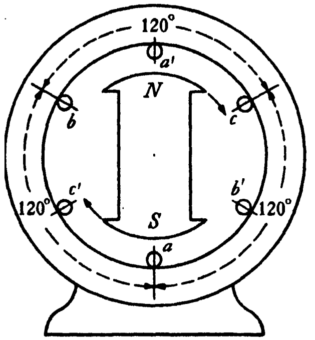


Fig. 4-1. Elementary 3-Phase Alternator.

In practice, the polyphase alternator is constructed with coils distributed over the entire surface of the armature as shown in Fig. 4-2. Coils which lie within the arcs *A* and *A′* are connected in series with one another to form phase group *a*, those which lie within the arcs *B* and *B′* form phase group *b*, and those which lie within arcs *C* and *C′* form phase group *c*. Although the emfs of the individual coils in the phase group are displaced from one another in time phase, the resultant emf of the group will have the same time phase as the coil which lies in the center of the group. If the emfs induced in the coils are assumed to be of sine-wave form, they could be traced as shown in Fig. 4-3.

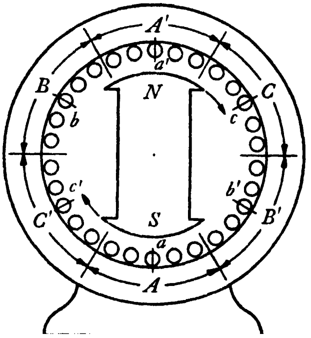


Fig. 4-2. Three-Phase Alternator with Coils Distributed Over Armature Surface.

A *2-phase* system can be obtained by winding the armature of an alternator with two independent phase groups so displaced as to cause one voltage to lag the other by 90 time-degrees. If the centers of the coils forming a 2-phase system are interconnected electrically, the system is called a *4-phase* or *quarter-phase* system. By properly spacing the windings of an alternator a polyphase system with any number of phases can be formed. For an *n-phase* system the displacement between consecutive phase groups would be $360/n$ time-degrees.

Of the various polyphase systems the 3-phase system is used most frequently, and because of this fact, this chapter will be chiefly devoted to 3-phase circuits. The *6-phase* system is used chiefly for operation of synchronous converters, and both 6- and 12-phase voltages are used in rectifier circuits.

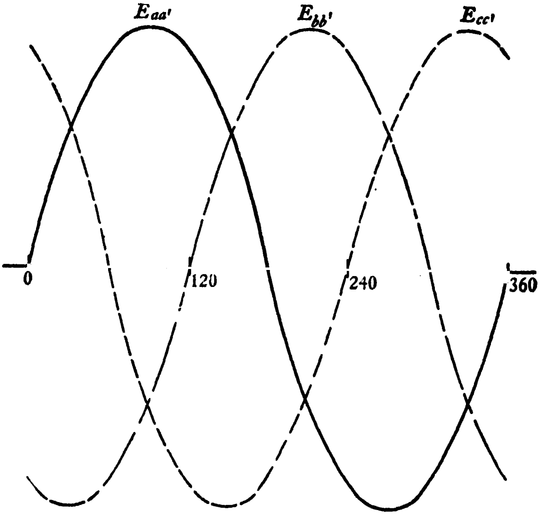


Fig. 4–3. Three-Phase Voltage Waves.

3. Double-Subscript Notation. Direct current is assumed to flow from positive to negative in the circuit external to the source of emf. D-c circuits are generally traced by starting at the positive terminal of the power source and following the circuit through to the negative. Thus, positive and negative signs serve to simplify the task of tracing d-c circuits. Alternating current reverses its direction periodically and the terms positive and negative, except as they refer to instantaneous values of voltage and current, cannot be used to refer to the potentials of fixed positions in an a-c circuit. Thus, a system of notation other than the customary positive and negative markings used for d-c circuits would be very convenient in designating the direction in which an a-c circuit is being traced. One such method of a-c circuit and vector notation is called the *double-subscript* method.

Consider the emfs of the two coils (Fig. 4-4*a*) to be displaced from one another by 120 time-degrees, and let it be required to connect these coils in series so that their emfs add at 120°. Before the series connection can be made, it is essential to know the directions in which the emfs are traced. Assuming that the emf as traced from *c* to *d* lags the emf as traced from *a* to *b* by 120°, coil end *b* could then be connected to coil end *c* to give the correct angular relationship for the series combination as specified. In double-subscript notation the symbol E_{ab} represents the emf as traced from *a* to *b*, and E_{cd} the emf from *c* to *d*. The resultant emf then is $E_{ad} = E_{ab} + E_{cd}$. The vector diagram (Fig. 4-4*b*) shows the relationship between the two coil emfs and their sum.

If the emfs of the two coils are traced from *d* to *a* after they have been combined in series with E_{cd} lagging E_{ab} by 120°, the resultant would be $E_{da} = E_{dc} + E_{ba}$. The vector diagram would then be constructed and labeled as shown in Fig. 4-4*c*. Attention is called to the fact that $-E_{cd} = E_{dc}$, $-E_{ab} = E_{ba}$, and $-E_{ad} = E_{da}$. That is, 180° displacement of a vector may be indicated either by interchanging the first and second subscripts of its symbol or by use of the negative sign. Since a symbol with double subscript notations represents a vector quantity, the dot which is usually placed under a symbol to identify it as a vector is no longer needed.

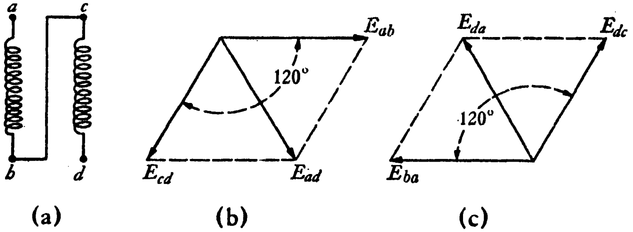


Fig. 4-4. Circuit and Vector Diagrams Illustrating Double-Subscript Notation.

Polyphase circuit problems generally involve several voltages and currents, and often appear to be complex. These problems may be greatly simplified: first, by drawing and labeling the circuit diagram, and second by constructing the vector diagram for the circuit with all voltages and currents indicated by double-subscript notation.

4. Phase Sequence. The order in which the phase voltages of a polyphase system pass through their positive maximum and other corresponding instantaneous values is called the *phase sequence* of the system. The phase sequence of the 3-phase voltages of the alternator (Fig. 4-1) is $E_{aa'}$, $E_{bb'}$, $E_{cc'}$ when the field is rotating in a clockwise direction. That is, the voltage wave $E_{aa'}$ is followed 120 time-degrees later by wave $E_{bb'}$ which in turn is followed 120 time-degrees later by wave $E_{cc'}$. The 3-phase voltage waves of Fig. 4-3 occur with this phase sequence, and the vectors which represent these waves are shown in Fig. 4-5*a*.

On the other hand, when the field rotation of the alternator (Fig. 4-1) is counterclockwise, the phase sequence of the voltages is $E_{aa'}$, $E_{cc'}$, $E_{bb'}$ as shown in Fig. 4-5*b*. [Attention is called to the fact that the voltage $E_{cc'}$ is plotted at − 30 time-degrees from the axis of reals in the vector diagram, while in the alternator (Fig. 4-1) the coil *cc'* is located − 150 space-degrees from the corresponding horizontal.] Likewise, $E_{bb'}$ is plotted − 150 time-degrees from the axis of reals, while coil *bb'* is drawn − 30 space-degrees from the corresponding horizontal. This is due to the fact that the vector diagram represents time-phase relations while space relations are used in arranging the coils on the armature of Fig. 4-1.

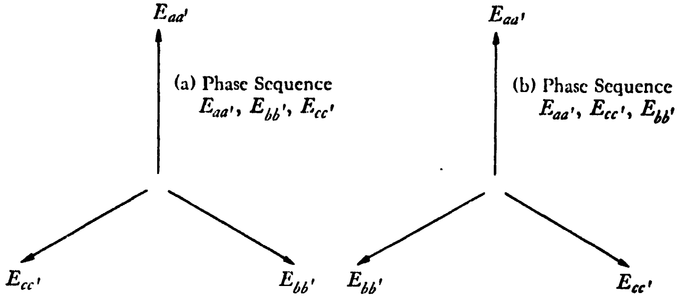


Fig. 4–5. Phase Sequence of 3-Phase Voltages.

The phase sequence of the voltages applied to a 3-phase load is fixed by the order in which the lines are connected to the load. By interchanging any pair of lines the phase sequence of the load is reversed. If the load is a 3-phase induction or synchronous motor, reversing the phase sequence causes the direction of rotation to be reversed. If the load is unbalanced, reversing the phase sequence will give different values to the line currents. Hence, in order to avoid confusion it is essential that the phase sequence be specified when calculating unbalanced 3-phase systems. Unless otherwise stated the phase sequence of 3-phase circuit problems presented in this

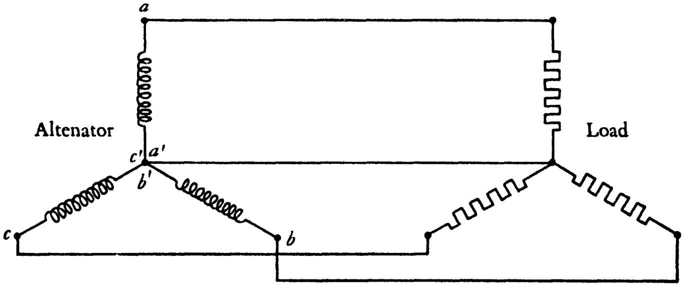
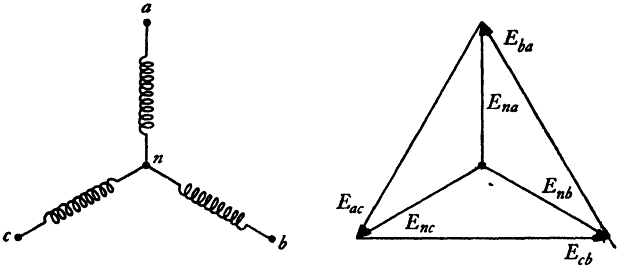


Fig. 4–6. Three-Phase, 4-Wire, Wye-Connected System.

chapter should be assumed as E_{oa}, E_{ob}, E_{oc} in order to check the answers given by the author.

5. Voltage Relations in Balanced Wye Systems. When the 3-coil groups of the alternator (Fig. 4-1) are electrically insulated from one another, 3 separate single-phase loads are required to load the 3 phases of the machine. In this case 6 wires would be needed for the connection between the alternator and the load. This type of 3-phase system is not used in practice. If coil ends a', b', and c' are connected together and one wire is brought out from this junction as shown in Fig. 4-6, the number of wires required between alternator and load would be reduced to four. This interconnection forms a *3-phase, 4-wire, wye* system. The three wires connected to coil ends a, b, and c are referred to as the *line* wires, and the fourth wire which connects to the junction of the 3-phase groups is called the *neutral* wire. Three-phase, 4-wire, wye systems are in general use and are particularly applicable in a-c network systems which supply single-phase lighting loads as well as 3-phase motor loads.



(a) (b)

Fig. 4-7. Circuit and Polygon-Type Diagrams of Voltages in a Balanced Wye System

When the currents in the 3-line wires of a wye system have equal magnitudes and are displaced from one another by 120 time-degrees the system is said to be *balanced.* Under balanced conditions no current flows in the neutral wire between the source and the load and the neutral wire can be removed without changing current and voltage values in the system. The system thus formed without the neutral wire is a *3-phase, 3-wire, wye* system. Three-phase, 3-wire, wye systems may be used for transmitting power to balanced loads. In general, 3-phase electrical equipment such as motors, heaters, etc. are constructed with 3 identical impedances, and if these impedances are wye connected they form a *balanced wye circuit.* It is customary practice to bring out 3 wires for connecting equipment of this type to the line.

The voltage between the neutral wire and each line wire (Fig. 4-6) is called a *phase voltage* and the voltage between any 2 line wires is called a *line voltage.* In a balanced wye system a definite relationship exists between the phase and line voltages. In order to form a wye system the phase groups must be combined so that the line voltage across any 2 phase groups equals the vector difference of the phase voltages, or in other words, the line voltage equals the phase voltages combined subtractively. Fig. 4-7 shows the relationship between phase and line voltage vectors in a wye system with a polygon-type diagram, while Fig. 4-8 shows the same relationship with a polar-type diagram. It is evident from either diagram that the line voltages,

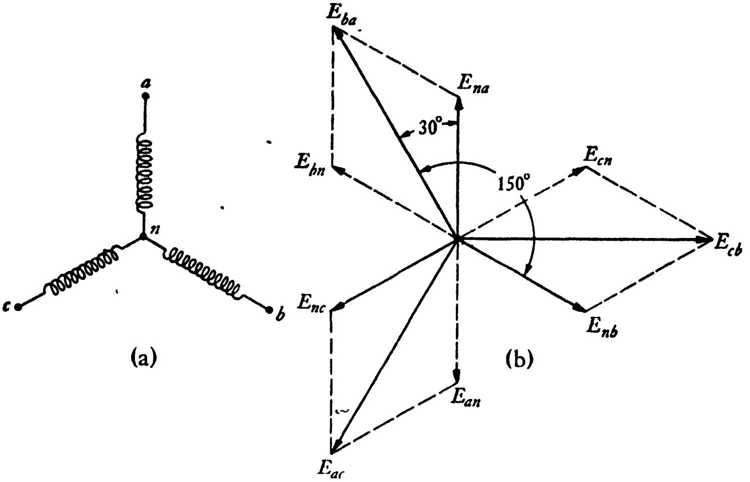


Fig. 4-8. Circuit and Polar-Type Diagrams of Voltages in a Balanced Wye System.

$$
\begin{aligned}
E_{ba} &= E_{bn} + E_{na}\\
E_{cb} &= E_{cn} + E_{nb}\\
E_{ac} &= E_{an} + E_{nc}
\end{aligned}
$$

For a balanced wye system the magnitude of the line voltage as determined by the law of cosines is,

$$E_{ba} = \sqrt{E_{nb}^2 + E_{na}^2 - 2E_{nb}E_{na}\cos 120°}$$

Or, if it is more apparent in Fig. 4-8

$$E_{ba} = \sqrt{E_{bn}^2 + E_{na}^2 + 2E_{bn}E_{na}\cos 60°}$$

If the line voltage and phase voltage are represented by E_L and E_P respectively, these symbols may be substituted in either of the two preceding equations to obtain,

$$E_L = \sqrt{3}\,E_P = 1.732E_P \tag{4-1}$$

Attention is called to the fact that in a balanced wye system the line voltage lags or leads one of its component phase voltages by 30° and the other by 150°. As to whether the line voltage lags or leads the component phase voltages depends on the direction in which the circuit is traced.

Example 4-1: In a balanced, 3-phase wye system the magnitude of the phase voltage is 120 volts. Determine the magnitude of the line voltage.

Solution.

$$E_L = 1.732E_P = 1.732 \times 120 = 208 \text{ volts} \qquad \textit{Ans.}$$

Example 4-2: The magnitude of the phase voltage of the balanced, 3-phase, wye system (Fig. 4-8) is 100 volts. Express each phase and line voltage in both polar and rectangular coordinates.

Solution.

$$\begin{aligned}
E_{na} &= 100\underline{/90^\circ} = 0 + j\,100 && \textit{Ans.}\\
E_{nb} &= 100\overline{\backslash 30^\circ} = 86.6 - j\,50 && \textit{Ans.}\\
E_{nc} &= 100\overline{\backslash 150^\circ} = -\,86.6 - j\,50 && \textit{Ans.}\\
E_{ba} &= E_{bn} + E_{na} = -\,86.6 + j\,50 + 0 + j\,100 &&\\
&= -\,86.6 + j\,150 = 173.2\underline{/120^\circ} && \textit{Ans.}\\
E_{cb} &= E_{cn} + E_{nb} = 86.6 + j\,50 + 86.6 - j\,50 &&\\
&= 173.2 + j\,0 = 173.2\underline{/0^\circ} && \textit{Ans.}\\
E_{ac} &= E_{an} + E_{nc} = 0 - j\,100 - 86.6 - j\,50 &&\\
&= -\,86.6 - j\,150 = 173.2\overline{\backslash 120^\circ} && \textit{Ans.}
\end{aligned}$$

Problem 4-1: The magnitude of the line voltage in a balanced, 3-phase, wye system is 4150 volts. Determine the magnitude of the phase voltage.

Problem 4-2: The magnitude of the line voltage of a balanced, 3-phase wye system is 480 volts. Draw the circuit and vector diagrams, placing E_{na} along the axis of reals and labeling the phase voltages in the sequence E_{na}, E_{nb}, E_{nc}. Express each phase and line voltage in both polar and rectangular coordinates.

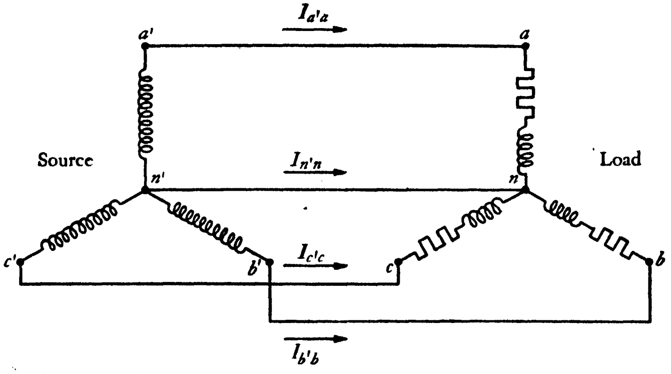


Fig. 4-9. Current Relations in a 4-Wire, Wye-Wye System.

6. Current Relations in Balanced Wye Systems. Fig. 4-9 shows a 3-phase, 4-wire, wye load connected to a 3-phase, 4-wire, wye source. It is evident that the line current $I_{a'a}$ is the same as the generator phase current $I_{n'a'}$ and the load phase current I_{an}. Also, $I_{b'b}$ is the same as $I_{n'b'}$ and I_{bn}, and $I_{c'c}$ is the same as $I_{n'c'}$ and I_{cn}. Thus, in a wye system $I_L = I_P$. The current in the neutral wire can be expressed in terms of the current in the 3 line wires by writing Kirchhoff's current law for the junction n.

$$I_{n'n} + I_{a'a} + I_{b'b} + I_{c'c} = 0$$

$$I_{nn'} = I_{a'a} + I_{b'b} + I_{c'c} \qquad (4\text{-}2)$$

When the load is balanced, the line currents have equal magnitudes and are displaced from one another by 120 time-degrees. Under this condition the vector sum of the line currents is zero, and no current flows in the neutral wire.

7. Voltage Relations in Balanced Delta Systems. If coil end a' (Fig. 4-1) is connected to b and coil end b' is connected to c, the emfs of the three coils will be combined additively with an angle of 120 time-degrees between any two coil emfs. The combined emf

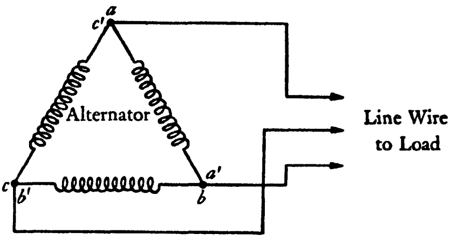
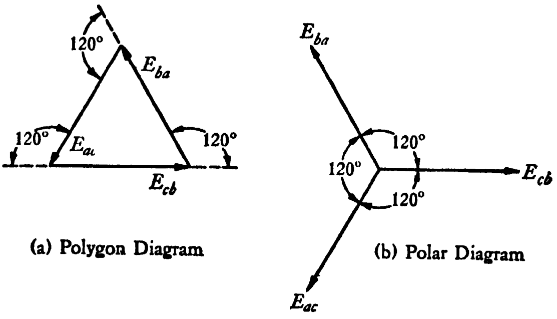


Fig. 4-10. Three-Phase, Delta-Connected Alternator.

$$E_{ac'} = E_{aa'} + E_{bb'} + E_{cc'}$$

When the magnitudes of the phase emfs are equal, and when the angle between any two emfs is 120 time-degrees, the value of $E_{ac'}$ will be zero and coil end a can be connected to c' as shown in Fig. 4-10. A lead is brought out for connection to the line from each junction of two coils. This method of interconnecting the coil groups of a 3-phase alternator forms a *delta* system.



(a) Polygon Diagram

(b) Polar Diagram

Fig. 4-11. Vector Diagrams of Voltages in a Balanced Delta System.

Fig. 4-11 shows the polygon and polar vector diagrams of the emfs for a delta-connected generator in which the emfs are balanced. It is evident that the line voltages and the phase voltages are the same in the delta connection, i.e., $E_L = E_P$.

8. Current Relations in Balanced Delta Systems. In Fig. 4-12 a delta load is shown connected across a delta source. The line current $I_{a'a}$ may be expressed in terms of the load phase currents I_{ab} and I_{ac} by writing Kirchhoff's current law about junction a. That is,

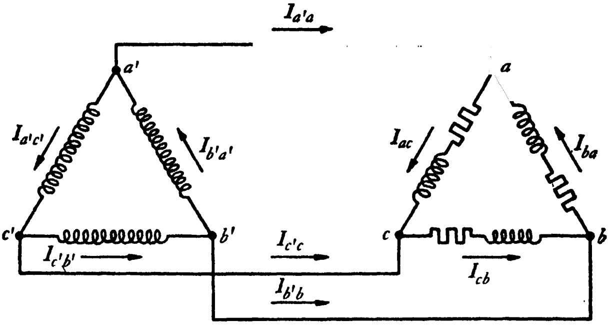


Fig. 4-12. Current Relations in a Delta System.

$$I_{a'a} + I_{ca} + I_{ba} = 0$$
$$I_{a'a} = I_{ac} + I_{ab}$$

In a similar manner line currents $I_{b'b}$ and $I_{c'c}$ may be written in terms of load phase currents as follows:

$$I_{b'b} = I_{ba} + I_{bc}$$
$$I_{c'c} = I_{cb} + I_{ca}$$

A definite relationship exists between the phase current and the line current in a balanced delta system. This relationship is shown by the polar vector diagram for a balanced delta load, Fig. 4-13. Phase current I_{cb} is assumed to lie along the axis of reals, and phase currents I_{ac} and I_{ba} follow in the order given, each displaced from the other by 120 time-degrees. Phase currents I_{ac} and I_{ba} are combined subtractively at 120° time phase to obtain $I_{a'a}$; I_{ba} and I_{cb} are combined subtractively at 120° to obtain $I_{b'b}$; and I_{cb} and I_{ac} are combined subtractively at 120° to obtain $I_{c'c}$. The magnitude of the line current in terms of the phase current as determined by the law of cosines is,

$$I_{a'a} = \sqrt{I_{ac}^2 + I_{ab}^2 + 2I_{ac}I_{ab}\cos 60^\circ}$$

If the line current and phase current are represented by the symbols I_L and I_P, respectively, these symbols may be substituted in the preceding equation to obtain,

$$I_L = \sqrt{3}\, I_P = 1.732 I_P \tag{4-3}$$

Attention is called to the fact that the line current in the balanced delta system leads or lags one of its component phase currents by 30° and the other by 150°. Here again the lead or lag relationship depends on the direction in which the circuit is traced.

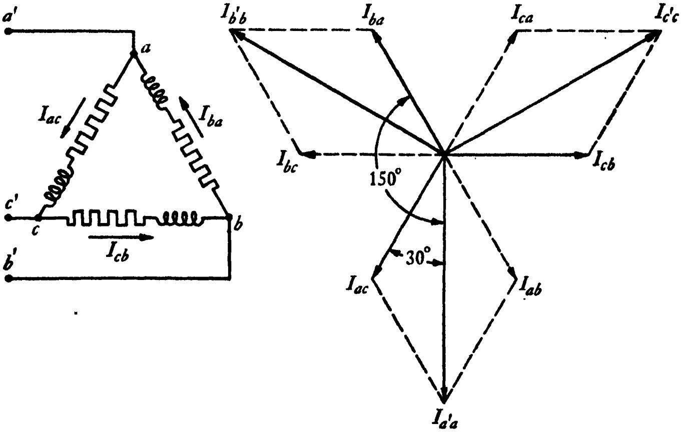


Fig. 4-13. Diagrams of Phase and Line Currents in a Balanced Delta System.

Three-phase delta-connected equipment, such as motors, heaters, etc., is constructed with 3 identical impedances, and, hence, constitutes a balanced delta load. Equation 4-3 is applicable to leads of this type.

Example 4-3: The phase current in a balanced delta system has a magnitude of 10 amp. Determine the magnitude of the line current.

Solution.

$$I_L = 1.732 I_P = 1.732 \times 10 = 17.32 \text{ amp.} \qquad \textit{Ans.}$$

Problem 4-3: The magnitude of the line current drawn by a 3-phase balanced delta load is 145 amp. Determine the magnitude of the phase current.

Problem 4-4: The 3 single-phase windings of a 3-phase alternator each have ratings of 2400 volts and 416 amperes. (*a*) Determine the rated line voltage and line current when the coils are wye connected; (*b*) when delta connected.

Problem 4-5: The low voltage coils of 3 single-phase transformers each have ratings of 120 volts and 625 amperes. (*a*) Determine the line voltage and line current ratings on the low voltage side of the 3-phase bank when the 3 low voltage coils are wye connected; (*b*) when delta connected.

9. Current in the Neutral of Unbalanced Wye Systems. When three impedances of different values are interconnected to form a 4-wire, wye system, phase currents of different magnitudes will flow in the system when a 3-phase voltage is applied. Also, if the ratios of resistance to reactance in the 3 impedances are different, the displacement angle between the phase currents will not be 120°. When either or both of these conditions exists, the system is *unbalanced* and current will flow in the neutral wire. The neutral current is given by equation 4-2 (see Fig. 4-9) in terms of line currents as

$$I_{nn'} = I_{a'a} + I_{b'b} + I_{c'c}$$

Expressed in phase current notations, equation 4-2 becomes

$$I_{nn'} = I_{an} + I_{bn} + I_{cn} \tag{4-4}$$

Example 4-4: The phase currents in a 4-wire, wye-connected load are as follows: $I_{an} = 10\underline{/0^\circ}$, $I_{bn} = 12\overline{\backslash 150^\circ}$, $I_{cn} = 8\underline{/165^\circ}$. Determine the current in the neutral wire.

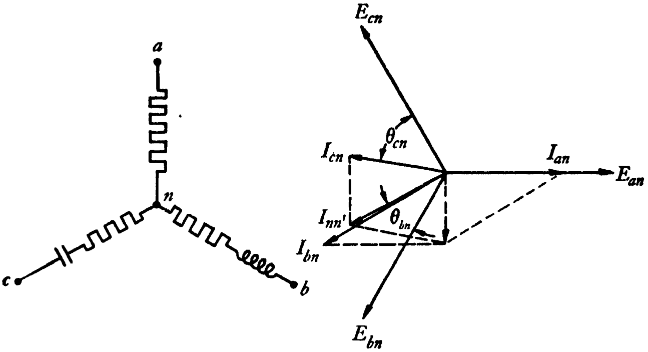


Fig. 4-14. Diagrams for Example 4-4.

Solution. See Fig. 4-14.

$$\begin{aligned}
I_{an} &= 10\underline{/0^\circ} = 10.00 + j\,0.00 \text{ amp}\\
I_{bn} &= 12\overline{\backslash 150^\circ} = -\,10.39 - j\,6.00 \text{ amp}\\
I_{cn} &= 8\underline{/165^\circ} = -\,7.76 + j\,1.94 \text{ amp}\\
I_{nn'} &= 10.00 + j\,0.00 - 10.39 - j\,6.00 - 7.76 + j\,1.94\\
&= -\,8.15 - j\,4.06\\
I_{nn'}\overline{\backslash\theta} &= \sqrt{(-\,8.15)^2 + (-\,4.06)^2}\,\overline{\backslash \tan^{-1} -4.06/-8.15}\\
&= 9.1\overline{\backslash 153^\circ\,30'} && \textit{Ans.}
\end{aligned}$$

Problem 4-6: The 3-phase currents in a 4-wire, wye-connected load are $I_{an} = 64\overline{\backslash 30^\circ}$, $I_{bn} = 52\overline{\backslash 135^\circ}$, $I_{cn} = 35\underline{/75^\circ}$. Assuming phase sequence to be ***a b c*** determine the current in the neutral wire.

Problem 4-7: In a 4-wire, wye-connected, unbalanced load, $I_{an} = 16\overline{\backslash 26^\circ}$, $I_{cn} = 18\overline{\backslash 140^\circ}$, and $I_{bn} = 14\underline{/150^\circ}$. Assuming the phase sequence to be ***a c b***, determine the neutral current.

10. Unbalanced 3-Wire, Wye-Wye Systems. Three unlike impedances Z_1, Z_2, and Z_3 are connected to a 3-wire, 3-phase generator as shown in Fig. 4-15. The impedance of each generator phase winding is represented by Z_G, and the impedance of each line wire between generator and load is represented by Z_L. The induced phase voltages of the generator are represented by $E_{n'a'}$, $E_{n'b'}$, and $E_{n'c'}$ with phase sequence in the order as stated. When the phase emfs, and the generator, the line, and the load impedances are known, the currents and voltages of the system may be computed by applying Kirchhoff's current and voltage laws.

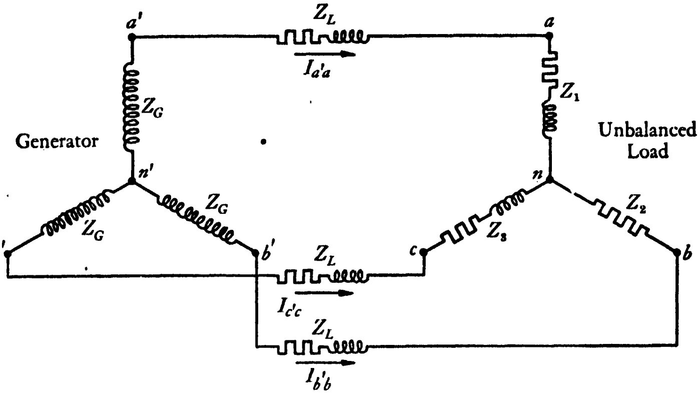


Fig. 4-15. Unbalanced, 3-Wire, Wye-Wye System.

The current equation at junction n (Fig. 4-15) is

$$I_{a'a} + I_{b'b} + I_{c'c} = 0 \tag{4-5}$$

Assuming one positive circuit direction for tracing to be n' a' a n b b' n', the voltage equation for this circuit is

$$E_{n'a'} - I_{a'a}(Z_G + Z_L + Z_1) - I_{bb'}(Z_G + Z_L + Z_2) - E_{n'b'} = 0 \tag{4-6}$$

Assuming the second positive circuit direction for tracing to be n' b' b n c c' n', the voltage equation for this circuit is

$$E_{n'b'} - I_{b'b}(Z_G + Z_L + Z_2) - I_{cc'}(Z_G + Z_L + Z_3) - E_{n'c'} = 0 \tag{4-7}$$

To simplify equations 4-6 and 4-7, set

$$Z_a = Z_G + Z_L + Z_1$$
$$Z_b = Z_G + Z_L + Z_2$$
$$Z_c = Z_G + Z_L + Z_3$$

Rearranging equations 4-6 and 4-7 with the symbols Z_a, Z_b, and Z_c, and at the same time substituting for $-I_{bb'}$ its value $I_{b'b}$, and for $-I_{cc'}$ its value $I_{c'c}$,

$$E_{n'a'} - E_{n'b'} - I_{a'a}Z_a + I_{b'b}Z_b = 0 \tag{4-8}$$
$$E_{n'b'} - E_{n'c'} - I_{b'b}Z_b + I_{c'c}Z_c = 0 \tag{4-9}$$

In terms of line emfs,

$$E_{n'a'} - E_{n'b'} = E_{b'n'} + E_{n'a'} = E_{b'a'}$$
$$E_{n'b'} - E_{n'c'} = E_{c'n'} + E_{n'b'} = E_{c'b'}$$

Substituting line emfs for phase emfs, equations 4-8 and 4-9 become

$$E_{b'a'} - I_{a'a}Z_a + I_{b'b}Z_b = 0 \tag{4-10}$$
$$E_{c'b'} - I_{b'b}Z_b + I_{c'c}Z_c = 0 \tag{4-11}$$

Substituting the equivalent of $I_{c'c} = (-I_{a'a} - I_{b'b})$ from equation 4-5 in equation 4-11 and solving for $I_{b'b}$,

$$E_{c'b'} - I_{b'b}Z_b - I_{b'b}Z_c - I_{a'a}Z_c = 0$$
$$I_{b'b} = \frac{E_{c'b'} - I_{a'a}Z_c}{Z_b + Z_c} \tag{4-12}$$

Solving for $I_{b'b}$ from equation 4-10,

$$I_{b'b} = \frac{-E_{b'a'} + I_{a'a}Z_a}{Z_b} \tag{4-13}$$

Equating the equivalents of $I_{b'b}$ in equations 4-12 and 4-13 and solving for $I_{a'a}$,

$$I_{a'a} = \frac{E_{c'b'}Z_b + E_{b'a'}(Z_b + Z_c)}{Z_bZ_c + Z_a(Z_b + Z_c)} \tag{4-14}$$

In a similar manner it can be shown that

$$I_{b'b} = \frac{E_{c'b'}Z_a - E_{b'a'}Z_c}{Z_bZ_c + Z_a(Z_b + Z_c)} \tag{4-15}$$
$$I_{c'c} = -I_{a'a} - I_{b'b} \tag{4-16}$$

Attention is called to the fact that the denominators of equations 4-14 and 4-15 are identical. That is,

$$D = Z_bZ_c + Z_a(Z_b + Z_c)$$

The phase voltages of the load are

$$V_{an} = I_{a'a}Z_1$$
$$V_{bn} = I_{b'b}Z_2$$
$$V_{cn} = I_{c'c}Z_3$$

Each line voltage at the load can be determined by combining subtractively the phase voltages which form its components. That is,

$$V_{ab} = V_{an} - V_{bn}$$
$$V_{bc} = V_{bn} - V_{cn}$$
$$V_{ca} = V_{cn} - V_{an}$$

Example 4-5: Given the following values for the circuit of Fig. 4-15, determine: (*a*) the line currents; (*b*) the phase voltages; (*c*) the line voltages; (*d*) the alternator impedance voltages; and (*e*) the line impedance voltages. Solve with phase sequence $E_{n'a'}$, $E_{n'b'}$, $E_{n'c'}$.

$$\begin{aligned}
E_{n'a'} &= 2400 + j\,0 = 2400\underline{/0^\circ} \text{ volts}\\
E_{n'b'} &= -\ 1200 - j\,2080 = 2400\overline{\backslash 120^\circ} \text{ volts}\\
E_{n'c'} &= -\ 1200 + j\,2080 = 2400\underline{/120^\circ} \text{ volts}\\
Z_G &= 1 + j\,4 = 4.13\underline{/75^\circ\ 58'} \text{ ohms}\\
Z_L &= 6 + j\,4 = 7.21\underline{/33^\circ\ 42'} \text{ ohms}\\
Z_1 &= 34.7 + j\,20.0 = 40\underline{/30^\circ} \text{ ohms}\\
Z_2 &= 50.0 + j\,0 = 50\underline{/0^\circ} \text{ ohms}\\
Z_3 &= 42.4 + j\,42.4 = 60\underline{/45^\circ} \text{ ohms}
\end{aligned}$$

Solution.

(*a*)

$$\begin{aligned}
Z_a &= Z_G + Z_L + Z_1 = 41.7 + j\,28.0 = 50.3\underline{/33^\circ\ 51'} \text{ ohms}\\
Z_b &= Z_G + Z_L + Z_2 = 57.0 + j\,8.0 = 57.6\underline{/8^\circ} \text{ ohms}\\
Z_c &= Z_G + Z_L + Z_3 = 49.4 + j\,50.4 = 70.6\underline{/45^\circ\ 34'} \text{ ohms}\\
E_{c'b'} &= 1200 - j\,2080 - 1200 - j\,2080\\
&= 0 - j\,4160 = 4160\overline{\backslash 90^\circ} \text{ volts}\\
E_{b'a'} &= 1200 + j\,2080 + 2400 - j\,0\\
&= 3600 + j\,2080 = 4160\underline{/30^\circ} \text{ volts}\\
Z_b + Z_c &= 57.0 + j\,8.0 + 49.4 + j\,50.4\\
&= 106.4 + j\,58.4 = 121.4\underline{/28^\circ\ 44'} \text{ ohms}
\end{aligned}$$

Solving for the denominator of equations 4-14 and 4-15,

$$\begin{aligned}
D = {} & (57.0 + j\,8.0)(49.4 + j\,50.4) + (41.7 + j\,28.0)\\
& (106.4 + j\,58.4) = 5217 + j\,8680 = 10{,}120\underline{/58^\circ\ 56'} \text{ ohms}^2
\end{aligned}$$

Solving for $I_{a'a}$, $I_{b'b}$, and $I_{c'c}$ by substituting in equations 4-14, 4-15 and 4-16,

$$\begin{aligned}
I_{a'a} &= \frac{(4160\overline{\backslash 90^\circ})(57.6\underline{/8^\circ}) + (4150\underline{/30^\circ})(121.4\underline{/28^\circ\ 44'})}{10{,}120\underline{/58^\circ\ 56'}}\\
&= \frac{353{,}500\underline{/33^\circ\ 17'}}{10{,}120\underline{/58^\circ\ 56'}} = 34.9\overline{\backslash 25^\circ\ 39'} \text{ amp} \qquad \textit{Ans.}\\
I_{b'b} &= \frac{(4160\overline{\backslash 90^\circ})(50.3\underline{/33^\circ\ 51'}) - (4160\underline{/30^\circ})(70.6\underline{/45^\circ\ 34'})}{10{,}120\underline{/58^\circ\ 56'}}\\
&= \frac{45{,}900\overline{\backslash 84^\circ\ 36'}}{10{,}120\underline{/58^\circ\ 56'}} = 45.3\overline{\backslash 143^\circ\ 32'} \text{ amp} \qquad \textit{Ans.}\\
I_{c'c} &= -\ (34.9\overline{\backslash 25^\circ\ 39'}) - (45.3\overline{\backslash 143^\circ\ 32'})\\
&= -\ 31.4 + j\,15.1 + 36.4 + j\,26.9\\
&= 5.0 + j\,42.0 = 42.3\underline{/83^\circ\ 13'} \text{ amp} \qquad \textit{Ans.}
\end{aligned}$$

(*b*)

$$\begin{aligned}
V_{an} &= I_{a'a}Z_1 = (34.9\overline{\backslash 25^\circ\ 39'})(40\underline{/30^\circ})\\
&= 1396\underline{/4^\circ\ 21'} = 1391 + j\,106 \text{ volts} \qquad \textit{Ans.}\\
V_{bn} &= I_{b'b}Z_2 = (45.3\overline{\backslash 143^\circ\ 32'})(50\underline{/0})\\
&= 2265\overline{\backslash 143^\circ\ 32'} = -\ 1820 - j\,1343 \text{ volts} \qquad \textit{Ans.}
\end{aligned}$$

$$\begin{aligned}
V_{cn} &= I_{c'c}Z_3 = (42.3\underline{/83^\circ\ 13'})(60\underline{/45^\circ}) \\
&= 2538\underline{/128^\circ\ 13'} = -\ 1568 + j\,1992 \text{ volts} && \textit{Ans.}
\end{aligned}$$

(*c*)

$$\begin{aligned}
V_{ab} &= V_{an} - V_{bn} = (1391 + j\,106) - (-\ 1820 - j\,1343) \\
&= 3211 + j\,1449 = 3520\underline{/24^\circ\ 17'} \text{ volts} && \textit{Ans.} \\
V_{bc} &= V_{bn} - V_{cn} = (-\ 1820 - j\,1343) - (-\ 1568 + j\,1992) \\
&= -\ 252 - j\,3335 = 3344\diagdown\overline{94^\circ\ 19'} \text{ volts} && \textit{Ans.} \\
V_{ca} &= V_{cn} - V_{an} = (-\ 1568 + j\,1992) - (1391 + j\,106) \\
&= -\ 2959 + j\,1886 = 3509\underline{/147^\circ\ 28'} \text{ volts} && \textit{Ans.}
\end{aligned}$$

(*d*)

$$\begin{aligned}
I_{a'a}Z_G &= (34.9\diagdown\overline{25^\circ\ 39'})(4.13\underline{/75^\circ\ 58'}) \\
&= 144\underline{/50^\circ\ 19'} \text{ volts} && \textit{Ans.} \\
I_{b'b}Z_G &= (45.3\diagdown\overline{143^\circ\ 32'})(4.13\underline{/75^\circ\ 58'}) \\
&= 187\diagdown\overline{67^\circ\ 34'} \text{ volts} && \textit{Ans.} \\
I_{c'c}Z_G &= (42.3\underline{/83^\circ\ 13'})(4.13\underline{/75^\circ\ 58'}) \\
&= 174.5\underline{/159^\circ\ 11'} \text{ volts} && \textit{Ans.}
\end{aligned}$$

(*e*)

$$\begin{aligned}
I_{a'a}Z_L &= (34.9\diagdown\overline{25^\circ\ 39'})(7.21\underline{/33^\circ\ 42'}) \\
&= 252\underline{/8^\circ\ 3'} \text{ volts} && \textit{Ans.} \\
I_{b'b}Z_L &= (45.3\diagdown\overline{143^\circ\ 32'})(7.21\underline{/33^\circ\ 42'}) \\
&= 326\diagdown\overline{109^\circ\ 50'} \text{ volts} && \textit{Ans.} \\
I_{c'c}Z_L &= (42.3\underline{/83^\circ\ 13'})(7.21\underline{/33^\circ\ 42'}) \\
&= 305\underline{/116^\circ\ 55'} \text{ volts} && \textit{Ans.}
\end{aligned}$$

The vector diagram (Fig. 4-16) is plotted from the given and computed data of example 4-5. The generator phase emfs $E_{n'a'}$, $E_{n'b'}$, and $E_{n'c'}$ are plotted from the intersection of the coordinate axes, with $E_{n'a'}$ on the axis of reals. The generator line emfs $E_{a'c'}$, $E_{b'a'}$, and $E_{c'b'}$ form an equilateral triangle which encloses the phase emfs. Generator, line and load impedance voltages for each path between n' and n (Fig. 4-15) are combined subtractively from the end of the respective phase emfs, and form the junction n on the vector diagram (Fig. 4-16). Attention is called to the fact that n does not fall on n', i.e., a difference in potential exists between the junction of the three generator phase windings and the junction of the three load impedances. The magnitude and direction of this voltage (not shown on diagram) could be represented by a vector connecting n' to n.

The voltages between line wires at the load terminals (not shown in Fig. 4-16) could be represented by vectors connecting the arrow heads of V_{an}, V_{bn}, and V_{cn}.

11. Unbalanced 4-wire, Wye-wye Systems. The voltages and currents in an unbalanced 4-wire, wye-wye system (Fig. 4-17) may be determined in a manner similar to the method previously described for the 3-wire, wye-wye system. The addition of the neutral wire introduces a fourth unknown current, $I_{nn'}$, and hence, four equations are required to determine the unknown values of voltage and current when the emfs and the impedances are known. The current equation at n is

$$I_{nn'} = I_{a'a} + I_{b'b} + I_{c'c} \tag{4-17}$$

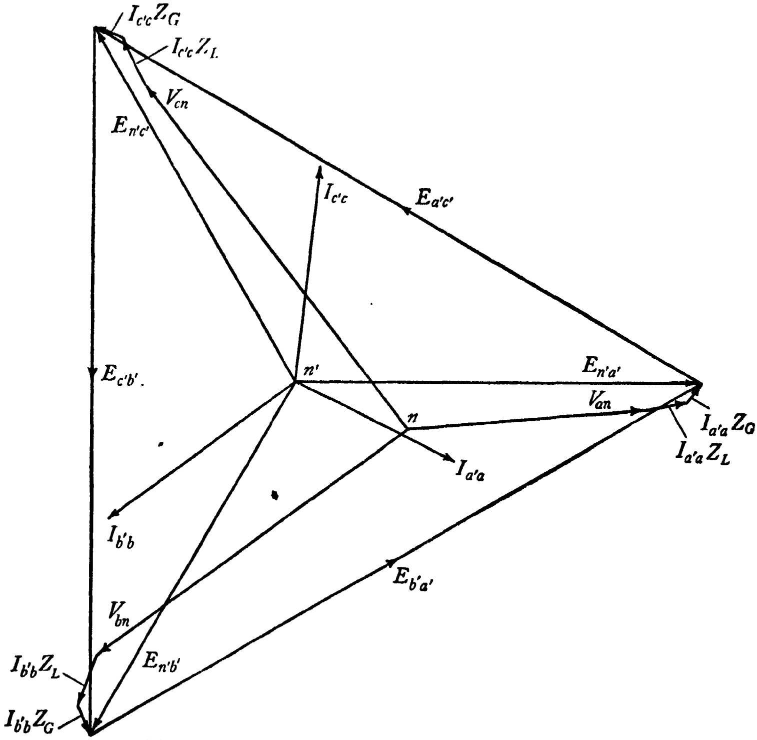


Fig. 4–16. Vector Diagram for Example 4–5.

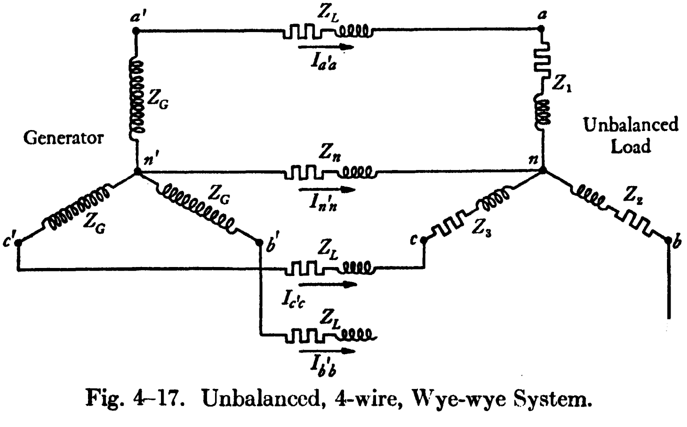


Fig. 4–17. Unbalanced, 4-wire, Wye-wye System.

With positive circuit direction, $n'\,a'\,a\,n\,n'$,

$$E_{n'a'} - I_{a'a}\,(Z_G + Z_L + Z_1) - I_{nn'}Z_n = 0 \tag{4-18}$$

With positive circuit direction, $n'\,b'\,b\,n\,n'$,

$$E_{n'b'} - I_{b'b}\,(Z_G + Z_L + Z_2) - I_{nn'}Z_n = 0 \tag{4-19}$$

With positive circuit direction, $n'\,c'\,c\,n\,n'$,

$$E_{n'c'} - I_{c'c}\,(Z_G + Z_L + Z_3) - I_{nn'}Z_n = 0 \tag{4-20}$$

To simplify equations 4-18, 4-19, and 4-20, set

$$\begin{aligned} Z_a &= Z_G + Z_L + Z_1 \\ Z_b &= Z_G + Z_L + Z_2 \\ Z_c &= Z_G + Z_L + Z_3 \end{aligned}$$

Substituting Z_a, Z_b, and Z_c in equations 4-18, 4-19, and 4-20, respectively, and at the same time expressing these equations in terms of the line currents,

$$I_{a'a} = \frac{E_{n'a'} - I_{nn'}Z_n}{Z_a} \tag{4-21}$$

$$I_{b'b} = \frac{E_{n'b'} - I_{nn'}Z_n}{Z_b} \tag{4-22}$$

$$I_{c'c} = \frac{E_{n'c'} - I_{nn'}Z_n}{Z_c} \tag{4-23}$$

The solution of equations 4-17, 4-21, 4-22, and 4-23 for either $I_{a'a}$, $I_{b'b}$, or $I_{c'c}$ in terms of emfs and impedances is an involved and laborious task. The neutral current $I_{nn'}$ may be determined with less involved procedure. Hence, the solution of unbalanced, 4-wire, wye-wye systems is simplified by first determining the numerical value of the neutral current for the particular case being considered, and then substituting this value of $I_{nn'}$ in equations 4-21, 4-22, and 4-23.

Substituting the values of $I_{a'a}$, $I_{b'b}$, and $I_{c'c}$ from equations 4-21, 4-22, and 4-23 into equation 4-17,

$$I_{nn'} = \frac{E_{n'a'} - I_{nn'}Z_n}{Z_a} + \frac{E_{n'b'} - I_{nn'}Z_n}{Z_b} + \frac{E_{n'c'} - I_{nn'}Z_n}{Z_c}$$

Solving for $I_{nn'}$ in terms of emfs and impedances,

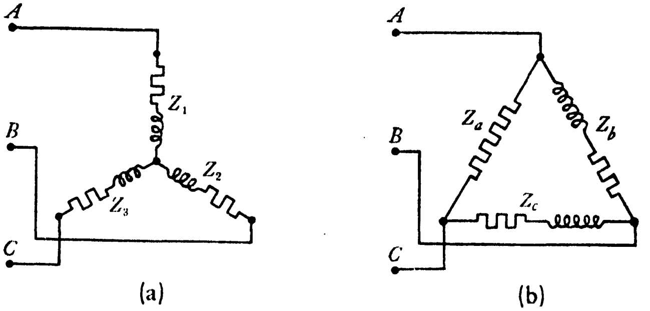
$$I_{nn'} = \frac{E_{n'a'}Z_bZ_c + E_{n'b'}Z_aZ_c + E_{n'c'}Z_aZ_b}{(Z_aZ_bZ_c) + Z_n\,(Z_bZ_c + Z_aZ_c + Z_aZ_b)} \tag{4-24}$$

Problem 4-8: The phase emfs of a three-wire, wye-connected generator have a magnitude of 6600 volts, and the impedance of each generator phase is $9\underline{/75}$ ohms. Three impedances $Z_1 = 70\underline{/20^\circ}$ ohms, $Z_2 = 60\overline{\backslash 15^\circ}$ ohms, and $Z_3 = 80\underline{/40^\circ}$ ohms are connected to form a 3-wire, wye-connected load across the line wires from the generator. The impedance of each line between the generator and the load terminals is $15\underline{/30^\circ}$ ohms. Z_1 is connected across E_{an}, Z_2 is connected across E_{bn}, and Z_3 across E_{cn}. Assume the phase sequence to be $E_{n'a'}$, $E_{n'b'}$, $E_{n'c'}$. Solve the problem with $E_{n'a'}$ considered to lie on the axis of reals. Determine (*a*) the current in each line; (*b*) the phase voltage across each leg of the load; (*c*) the line voltage at the load terminals; (*d*) the generator phase impedance voltages; and (*e*) the line impedance voltages. Construct a vector diagram similar to Fig. 4–16.

Problem 4-9: Assume the following values for the unbalanced, 4-wire, wye-wye system of Fig. 4-17 and determine: (*a*) the current in the neutral; (*b*) the current in each line wire; (*c*) the phase voltages at the terminals of the load; (*d*) the generator impedance voltage for each phase; (*e*) the line impedance voltage for each phase. Construct a vector diagram with the values given and obtained which will be similar to the vector diagram of Fig. 4-16. A vector from *n* to *n'* will represent the impedance voltage of the neutral wire.

$$
\begin{aligned}
E_{n'a'} &= 120 + j\,0 \text{ volts} \\
E_{n'b'} &= -\ 60 - j\,104 \text{ volts} \\
E_{n'c'} &= -\ 60 + j\,104 \text{ volts} \\
Z_G &= 0.02 + j\,0.05 \text{ ohms} \\
Z_L &= 0.06 + j\,0.04 \text{ ohms} \\
Z_n &= 0.12 + j\,0.04 \text{ ohms} \\
Z_1 &= 0.5 + j\,0.5 \text{ ohms} \\
Z_2 &= 0.6 - j\,0.2 \text{ ohms} \\
Z_3 &= 0.8 + j\,0 \text{ ohms.}
\end{aligned}
$$

12. Wye Equivalent of a Delta-connected System. The wye-connected system (Fig. 4-18*a*) is equivalent to the delta-connected system (Fig. 4-18*b*) when the impedance between any pair of corresponding line wires for both systems are the same, the third line wire being disconnected. With line wire *A* disconnected, the impedance between *B* and *C* is as follows: For the wye system,



(a) (b)

Fig. 4–18. Equivalent Wye and Delta Systems of Impedances.

$$Z_2 + Z_3$$

For the delta system,

$$\frac{Z_b\,(Z_a + Z_c)}{Z_a + Z_b + Z_c}$$

And when the systems are equivalent,

$$Z_2 + Z_3 = \frac{Z_c(Z_a + Z_b)}{Z_a + Z_b + Z_c} \tag{4-25}$$

For equivalent wye and delta systems with line wire B disconnected,

$$Z_1 + Z_3 = \frac{Z_a(Z_b + Z_c)}{Z_a + Z_b + Z_c} \tag{4-26}$$

and with line wire C disconnected,

$$Z_2 + Z_1 = \frac{Z_b(Z_a + Z_c)}{Z_a + Z_b + Z_c} \tag{4-27}$$

Solving equations 4-25, 4-26, and 4-27 simultaneously for Z_1, Z_2, and Z_3 gives

$$Z_1 = \frac{Z_aZ_b}{Z_a + Z_b + Z_c} \tag{4-28}$$

$$Z_2 = \frac{Z_bZ_c}{Z_a + Z_b + Z_c} \tag{4-29}$$

$$Z_3 = \frac{Z_aZ_c}{Z_a + Z_b + Z_c} \tag{4-30}$$

When the delta system is balanced $Z_a = Z_b = Z_c$. Representing each of the three equal impedances of the delta system by Z_Δ, and the impedance of each phase of the equivalent wye system by Z_Y, equations 4-25, 4-26, and 4-27 for a balanced system become

$$Z_Y = \frac{Z_\Delta^2}{3Z_\Delta} = \frac{Z_\Delta}{3} \tag{4-31}$$

13. Delta Equivalent of a Wye-Connected System. The delta equivalent of a wye-connected system may be obtained by solving equations 4-25, 4-26, and 4-27 simultaneously for Z_a, Z_b, and Z_c. This procedure gives

$$Z_a = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2} \tag{4-32}$$

$$Z_b = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3} \tag{4-33}$$

$$Z_c = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1} \tag{4-34}$$

When the wye system is balanced $Z_1 = Z_2 = Z_3$. Representing each phase impedance of the wye system by Z_Y and each of the three impedances of the equivalent delta system by Z_Δ, equations 4-32, 4-33, and 4-34 for a balanced system become

$$Z_\Delta = \frac{3Z_Y^2}{Z_Y} = 3Z_Y \tag{4-35}$$

Problem 4-10: Three impedances, each equal to $20 + j\,15$ ohms, are interconnected to form a balanced delta system. Determine the equivalent wye impedance for each phase of the wye.

Problem 4-11: The impedances of the delta-connected system (Fig. 4-18*b*) are $Z_a = 12 + j\,0$ ohms, $Z_b = 8 + j\,6$ ohms, and $Z_c = 10 + j\,5$ ohms. Determine the equivalent wye impedances Z_1, Z_2, and Z_3 as shown in Fig. 4-18*a*.

Problem 4-12: Three impedances, each equal to $1.5 - j\,12$ ohms, are interconnected to form a balanced wye system. Determine the impedance for each leg of an equivalent delta system.

Problem 4-13: The impedances of a wye-connected system are $50 + j\,20$ ohms, $40 - j\,30$ ohms, and $60 + j\,0$ ohms. Determine the impedances of an equivalent delta circuit, and sketch the two systems showing the relationships of the wye and delta impedances in respect to the 3-line wires.

14. Other Unbalanced 3-phase Systems. The computation of currents and voltages in unbalanced wye-delta, delta-delta, and delta-wye systems by the Kirchhoff law method is impracticable because of the great amount of work involved in solving six or more simultaneous equations. Systems of these types are usually solved by first converting them into equivalent 3-wire wye-wye systems, and then proceeding with the method given in article 10.

15. Unbalanced Delta Systems — Fixed Voltages Assumed at the Load Terminals. In practice, the generator and line impedance voltages in 3-phase systems are frequently small enough to be neglected. When this is done, balanced voltages having fixed values may be assumed between lines at the terminals of the load. This assumption greatly reduces the work required to compute current values.

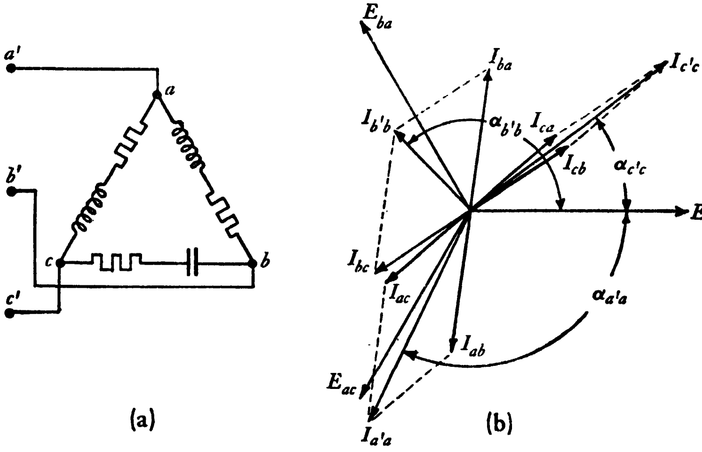


Fig. 4–19. Diagrams for Example 4–6; Phase Sequence E_{cb}, E_{ac}, E_{ba}.

For a balanced delta system, $I_L = \sqrt{3}\,I_P$ (equation 4-3). No definite relationship exists between line and phase currents in an unbalanced delta system. However, if the phase currents are known, or if they can be determined from line voltages at the terminals of the load which are assumed to be fixed and the load impedances, the line currents in an unbalanced delta system can be determined in a simple manner.

Example 4-6: The voltage impressed on the unbalanced delta system of impedances (Fig. 4-19*a*) has a fixed value of 220 volts between line wires and is assumed to be balanced. $Z_1 = 10 - j\,7$ ohms, $Z_2 = 12 + j\,4$ ohms, and $Z_3 = 8 + j\,6$ ohms. Determine the phase and line currents, first with phase sequence E_{cb}, E_{ac}, E_{ba} and then with phase sequence E_{cb}, E_{ba}, E_{ac}.

Solution. Phase sequence E_{cb}, E_{ac}, E_{ba}, with E_{cb} assumed to lie on the axis of reals. See Fig. 4-19.

$$\begin{aligned}
E_{cb} &= 220 + j\,0 \\
I_{cb} &= \frac{220 + j\,0}{10 - j\,7} = \frac{220 + j\,0}{10 - j\,7}\cdot\frac{10 + j\,7}{10 + j\,7} \\
&= \frac{2200 + j\,1540}{149} = 14.76 + j\,10.33 \text{ amp} \\
E_{ac} &= 220\,(\cos 240^\circ + j \sin 240^\circ) = -\,110 - j\,190.5 \\
I_{ac} &= \frac{-\,110 - j\,190.5}{12 + j\,4} = \frac{-\,110 - j\,190.5}{12 + j\,4}\cdot\frac{12 - j\,4}{12 - j\,4} \\
&= \frac{-\,2082 - j\,1846}{160} = -\,13.02 - j\,11.53 \text{ amp} \\
E_{ba} &= 220\,(\cos 120^\circ + j \sin 120^\circ) = -\,110 + j\,190.5 \\
I_{ba} &= \frac{-\,110 + j\,190.5}{8 + j\,6} = \frac{-\,110 + j\,190.5}{8 + j\,6}\cdot\frac{8 - j\,6}{8 - j\,6} \\
&= \frac{263 + j\,2184}{100} = 2.63 + j\,21.84 \text{ amp} \\
I_{a'a} &= I_{ac} + I_{ab} = -\,13.02 - j\,11.53 - 2.63 - j\,21.84 \\
&= -\,15.65 - j\,33.37 \text{ amp} && \textit{Ans.} \\
I_{a'a}\underline{/\alpha_{a'a}} &= \sqrt{(-\,15.65)^2 + (-\,33.37)^2}\,\underline{/\tan^{-1} -\,33.37/-\,15.65} \\
&= 36.86\overline{\backslash 115^\circ\,8'} \text{ amp} && \textit{Ans.} \\
I_{b'b} &= I_{ba} + I_{bc} = 2.63 + j\,21.84 - 14.76 - j\,10.33 \\
&= -\,12.13 + j\,11.55 \text{ amp} && \textit{Ans.} \\
I_{b'b}\underline{/\alpha_{b'b}} &= \sqrt{(-\,12.13)^2 + (11.55)^2}\,\underline{/\tan^{-1} 11.55/-\,12.13} \\
&= 16.8\underline{/133^\circ\,36'} \text{ amp} && \textit{Ans.} \\
I_{c'c} &= I_{cb} + I_{ca} = 14.76 + j\,10.33 + 13.02 + j\,11.53 \\
&= 27.78 + j\,21.86 \text{ amp} && \textit{Ans.} \\
I_{c'c}\underline{/\alpha_{c'c}} &= \sqrt{27.78^2 + 21.86^2}\,\underline{/\tan^{-1} 21.86/27.78} \\
&= 35.35\underline{/38^\circ\,17'} \text{ amp} && \textit{Ans.}
\end{aligned}$$

Solution. Phase sequence E_{cb}, E_{ba}, E_{ac}, with E_{cb} assumed to lie on the axis of reals. See diagrams, Fig. 4-20.

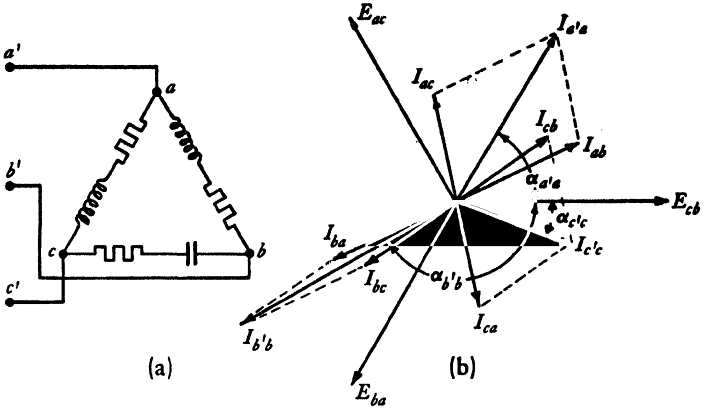


Fig. 4–20. Diagrams for Example 4–6; Phase Sequence E_{cb}, E_{ba}, E_{ac}.

$$E_{cb} = 220 + j\,0 \text{ volts}$$

$$I_{cb} = \frac{220 + j\,0}{10 - j\,7} = \frac{220 + j\,0}{10 - j\,7} \cdot \frac{10 + j\,7}{10 + j\,7}$$

$$= \frac{2200 + j\,1540}{149} = 14.76 + j\,10.33 \text{ amp}$$

$$E_{ba} = 220\,(\cos 240^\circ + j \sin 240^\circ) = -\,110 - j\,190.5 \text{ volts}$$

$$I_{ba} = \frac{-\,110 - j\,190.5}{8 + j\,6} = \frac{-\,110 - j\,190.5}{8 + j\,6} \cdot \frac{8 - j\,6}{8 - j\,6}$$

$$= \frac{-\,1943 - j\,924}{100} = -\,19.43 - j\,9.24 \text{ amp}$$

$$E_{ac} = 220\,(\cos 120^\circ + j \sin 120^\circ) = -\,110 + j\,190.5 \text{ volts}$$

$$I_{ac} = \frac{-\,110 + j\,190.5}{12 + j\,4} = \frac{-\,110 + j\,190.5}{12 + j\,4} \cdot \frac{12 - j\,4}{12 - j\,4}$$

$$= \frac{-\,558 + j\,2726}{160} = -\,3.49 + j\,17.04 \text{ amp}$$

$$I_{a'a} = I_{ac} + I_{ab} = -\,3.49 + j\,17.04 + 19.43 + j\,9.24$$

$$= 15.94 + j\,26.28 \text{ amp}$$ *Ans.*

$$I_{a'a}\underline{/\alpha_{a'a}} = \sqrt{15.94^2 + 26.28^2}\;\underline{/\tan^{-1} 26.28/15.94}$$

$$= 30.7\underline{/58^\circ\,46'} \text{ amp}$$ *Ans.*

$$I_{b'b} = I_{ba} + I_{bc} = -\,19.43 - j\,9.24 - 14.76 - j\,10.33$$

$$= -\,34.19 - j\,19.57 \text{ amp}$$ *Ans.*

$$I_{b'b}\underline{/\alpha_{b'b}} = \sqrt{(-\,34.19)^2 + (-\,19.57)^2}\;\underline{/\tan^{-1}(-\,19.57)/(-\,34.19)}$$

$$= 39.4\overline{\backslash 150^\circ\,11'} \text{ amp}$$ *Ans.*

$$I_{c'c} = I_{cb} + I_{ca} = 14.76 + j\,10.33 + 3.49 - j\,17.04$$

$$= 18.25 - j\,6.71 \text{ amp}$$ *Ans.*

$$I_{c'c}\underline{/\alpha_{c'c}} = \sqrt{18.25^2 + (-\,6.71)^2}\;\underline{/\tan^{-1}(-\,6.71)/18.25}$$

$$= 19.6\overline{\backslash 20^\circ\,12'} \text{ amp}$$ *Ans.*

Problem 4-14: The 3 bus bars leading from a bank of delta-connected transformers supply three 440-volt, 60-cycle single-phase loads. The first is connected across busses *a* and *c*, and draws 120 amp at 80 per cent lagging power factor. The second, connected across busses *b* and *a*, draws 100 amp at 90 per cent leading power factor, and the third, connected across busses *c* and *b*, draws 140 amp at unity power factor. Determine the current in each of the three busses. Assume the bus voltage to be balanced, to be fixed at 440 volts, and the phase sequence to be E_{ba}, E_{ac}, E_{cb}. Solve by placing E_{ac} on the axis of reals.

Problem 4-15: Three impedances $Z_1 = 40 + j\,16$, $Z_2 = 28 + j\,24$, and $Z_3 = 35 + j\,21$ are connected to form a delta load across the voltages E_{ac}, E_{ba}, and E_{cb} respectively. Assume the voltage to be balanced and to have a fixed magnitude of 2200 volts. Determine the current in the 3 lines feeding this load. The phase sequence is E_{ac}, E_{cb}, E_{ba}. Solve by placing E_{ba} along the axis of reals.

16. Combination 3-phase Systems.

Usually a generator supplies power to several wye and delta loads at the same time. In addition, several single-phase loads may be connected to the same source. A single-phase load on a 3-wire, 3-phase system would be connected between a pair of line wires, and, hence, would be similar to one leg of a delta system. When loads of different types are connected to a 3-wire, 3-phase source, and when the voltages at the terminals of the loads can be considered fixed, a simple method of determining the currents in the system with voltages and impedances known is to convert all wye-connected impedances to equivalent delta impedances. The phase currents in each load are then computed and the line currents are obtained by use of Kirchhoff's current law.

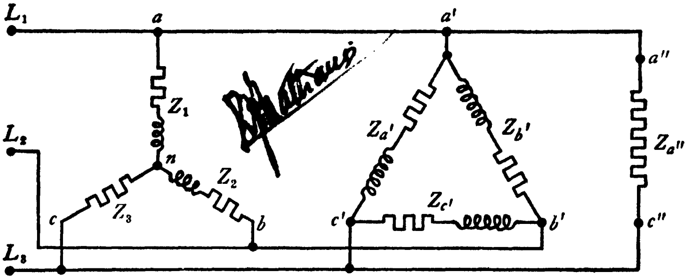


Fig. 4-21. Circuit Diagram for Example 4-7.

Example 4-7: Determine the current in each line wire for the combined load of Fig. 4-21. Consider the line voltages at the terminals of the load to be fixed and the phase sequence to be E_{ba}, E_{cb}, E_{ac}. The voltage and impedance values are as follows:

$$E_{ba} = 2300 + j\,0 = 2300\underline{/0^\circ} \text{ volts}$$
$$E_{cb} = -\ 1150 - j\,1992 = 2300\overline{\backslash 120^\circ} \text{ volts}$$
$$E_{ac} = -\ 1150 + j\,1992 = 2300\underline{/120^\circ} \text{ volts}$$

$$
\begin{aligned}
Z_1 &= 40 + j\,20 = 44.72\underline{/26^\circ\ 34'} \text{ ohms}\\
Z_2 &= 30 + j\,50 = 58.31\underline{/59^\circ\ 3'} \text{ ohms}\\
Z_3 &= 60 + j\,0 = 60\underline{/0^\circ} \text{ ohms}\\
Z_{a'} &= 80 + j\,10 = 80.62\underline{/7^\circ\ 8'} \text{ ohms}\\
Z_{b'} &= 90 + j\,30 = 94.87\underline{/18^\circ\ 26'} \text{ ohms}\\
Z_{c'} &= 70 + j\,70 = 98.99\underline{/45^\circ} \text{ ohms}\\
Z_{a''} &= 100 + j\,0 = 100\underline{/0^\circ} \text{ ohms}
\end{aligned}
$$

Solution. See vector diagram Fig. 4-22.

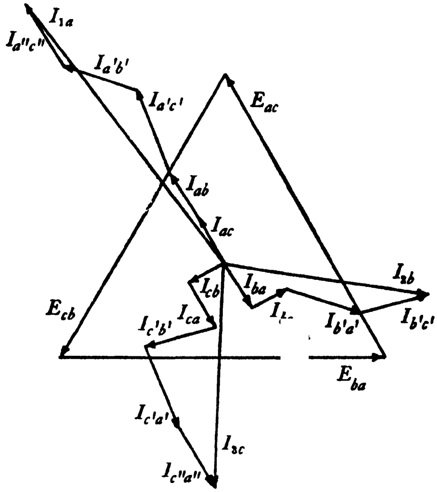


Fig. 4-22. Vector Diagram for Example 4-7.

Converting wye impedances Z_1, Z_2, and Z_3 to equivalent delta impedances.

$$
\begin{aligned}
Z_a &= \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}\\
&= \frac{4400 + j\,6800}{30 + j\,50} = 138.8 - j\,4.71 \text{ ohms}\\
Z_b &= \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}\\
&= \frac{4400 + j\,6800}{60 + j\,0} = 73.3 + j\,113.3 \text{ ohms}\\
Z_c &= \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}\\
&= \frac{4400 + j\,6800}{40 + j\,20} = 156.0 + j\,92.0 \text{ ohms}\\
I_{ba} &= \frac{E_{ba}}{Z_b} = \frac{2300 + j\,0}{73.3 + j\,113.3} = 9.30 - j\,14.40 \text{ amp}
\end{aligned}
$$

$$I_{cb} = \frac{E_{cb}}{Z_c} = \frac{-1150 - j\,1992}{156.0 + j\,92} = -\;11.05 - j\,6.25 \text{ amp}$$

$$I_{ac} = \frac{E_{ac}}{Z_a} = \frac{-1150 + j\,1992}{138.8 - j\,4.71} = -\;8.76 + j\,14.05 \text{ amp}$$

$$I_{b'a'} = \frac{E_{ba}}{Z_{b'}} = \frac{2300 + j\,0}{90 + j\,30} = 23.00 - j\,7.67 \text{ amp}$$

$$I_{c'b'} = \frac{E_{cb}}{Z_{c'}} = \frac{-1150 - j\,1992}{70 + j\,70} = -\;22.41 - j\,6.01 \text{ amp}$$

$$I_{a'c'} = \frac{E_{ac}}{Z_{a'}} = \frac{-1150 + j\,1992}{80 + j\,10} = -\;11.10 + j\,26.25 \text{ amp}$$

$$I_{a''c''} = \frac{E_{ac}}{Z_{a''}} = \frac{-1150 + j\,1992}{100 + j\,0} = -\;11.50 + j\,19.92 \text{ amp}$$

$$I_{1a} = I_{ac} + I_{ab} + I_{a'c'} + I_{a'b'} + I_{a''c''}$$
$$= -\;63.66 + j\,82.29 = 104.1\underline{/127^\circ\,47'} \text{ amp} \qquad \textit{Ans.}$$

$$I_{2b} = I_{ba} + I_{bc} + I_{b'a'} + I_{b'c'}$$
$$= 65.76 - j\,9.81 = 66.5\overline{\backslash 8^\circ\,29'} \text{ amp} \qquad \textit{Ans.}$$

$$I_{3c} = I_{cb} + I_{ca} + I_{c'b'} + I_{c'a'} + I_{c''a''}$$
$$= -\;2.10 - j\,71.78 = 71.8\overline{\backslash 91^\circ\,40'} \text{ amp} \qquad \textit{Ans.}$$

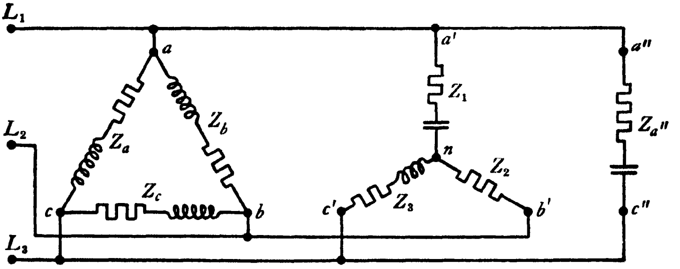


Fig. 4–23. Circuit for Problem 4–16.

Problem 4-16: Determine the current in each line wire for the combined load of Fig. 4-23 with fixed voltages assumed across the terminals of the loads. Solve for phase sequence E_{ba}, E_{cb}, E_{ac}. The voltages and impedances are as follows:

$$E_{ba} = 460 + j\,0 = 460\underline{/0^\circ} \text{ volts}$$
$$E_{cb} = -\;230 - j\,399 = 460\overline{\backslash 120^\circ} \text{ volts}$$
$$E_{ac} = -\;230 + j\,399 = 460\underline{/120^\circ} \text{ volts}$$
$$Z_a = 8 + j\,6 = 10.00\underline{/36^\circ\,52'} \text{ ohms}$$
$$Z_b = 9 + j\,2 = 9.22\underline{/12^\circ\,31'} \text{ ohms}$$
$$Z_c = 12 + j\,0 = 12.00\underline{/0^\circ} \text{ ohms}$$
$$Z_1 = 15 - j\,10 = 18.03\overline{\backslash 33^\circ\,42'} \text{ ohms}$$
$$Z_2 = 16 + j\,8 = 17.89\underline{/26^\circ\,34'} \text{ ohms}$$
$$Z_3 = 20 + j\,0 = 20.00\underline{/0^\circ} \text{ ohms}$$
$$Z_{a''} = 10 - j\,3 = 10.44\overline{\backslash 16^\circ\,42'} \text{ ohms}$$

17. Power in 3-phase Systems. The total power in a 3-phase system, whether balanced or unbalanced, is expressed by the equation

$$P_T = P_1 + P_2 + P_3 \tag{4-36}$$

where P_T is the 3-phase power and P_1, P_2, and P_3 each represent the single-phase power in the respective phases.

Under balanced conditions, $P_1 = P_2 = P_3$, and if each is represented by the symbol P_P, equation 4-36 may be written

$$P_T = 3P_P \tag{4-37}$$

The power in each phase of the balanced system expressed in terms of phase voltage, phase current, and phase angle is

$$P_P = E_P I_P \cos\theta$$

and by substitution,

$$P_T = 3E_P I_P \cos\theta \tag{4-38}$$

For a balanced wye-connected load, E_P equals the line voltage E_L divided by $\sqrt{3}$, and I_P equals the line current I_L. Rewriting equation 4-38 in terms of line values, we have

$$P_T = 3\frac{E_L}{\sqrt{3}} I_L \cos\theta$$

$$P_T = \sqrt{3}\, E_L I_L \cos\theta$$

For a balanced delta-connected load, $E_P = E_L$ and $I_P = I_L/\sqrt{3}$. Substituting these equivalent line values for phase values, equation 4-38 becomes

$$P_T = 3E_L\frac{I_L}{\sqrt{3}} \cos\theta$$

$$P_T = \sqrt{3}\, E_L I_L \cos\theta$$

Hence, the equation for 3-phase power in a balanced system, whether wye or delta-connected, is

$$P_T = \sqrt{3}\, E_L I_L \cos\theta \tag{4-39}$$

Example 4-8: A 3-phase balanced load draws 40 amp per terminal when connected to a 440-volt 3-phase system. The power factor of the load is 80 per cent lagging. Determine the power drawn by the load.

Solution.

$$P_T = 1.732 \times 440 \times 40 \times 0.8 = 30{,}483 \text{ watts.} \qquad \textit{Ans.}$$

Problem 4-17: The input power to a 3-phase induction motor is 76.2 kw. The voltage across the line terminals of the motor is 2250 volts and the current per terminal is 26.1 amp. Determine the power factor of the motor under these conditions.

Problem 4-18: A 3-phase, 25-cycle induction motor is rated at 30 horsepower (output) and 440 volts. At rated load its efficiency is given as 0.87 and its power factor is given as 0.92. Determine the rated current per terminal.

Problem 4-19: The 6 leads from the phase groups of a 3-phase alternator are brought out so that the machine may be connected either wye or delta. The rated phase voltage is 6600 volts and the rated phase current is 379 amp. Determine the rated line voltage, the rated line current, and the rated kw at 80 per cent power factor when the alternator is connected (*a*) wye; (*b*) delta.

18. Measurement of Power in 3-phase Systems. The power in a balanced or unbalanced 3-phase system may be measured with 3 single-phase wattmeters or with one 3-element polyphase wattmeter. Fig. 4-24 shows 3 single-phase wattmeters connected to measure power in a 4-wire wye circuit. Each wattmeter measures the power of one phase, and hence, the sum of their readings equals the total power in the circuit. This connection may also be used to measure power in a wye circuit when the neutral wire between the source and the load is omitted.

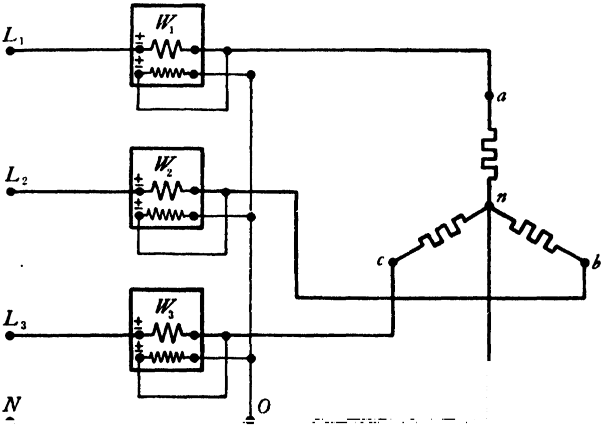


Fig. 4-24. Three Single-phase Wattmeters Connected to Measure Power in a 4-wire, Wye System.

Fig. 4-25 shows the connection of 3 single-phase wattmeters for measuring power in a delta circuit. Again, each wattmeter measures the power in one phase and the total power equals the sum of the values read by the 3 instruments.

The connection illustrated in Fig. 4-24 shows one end of each wattmeter potential element connected to the neutral of the circuit. This connection can be made easily when a neutral wire is used between the source and the load. However, neutral wires are seldom brought out from 3-phase, wye-connected equipment, and the connection of Fig. 4-24 is not practicable for circuits of this type. The connection shown in Fig. 4-25 requires that the current coil of each instrument be connected inside the delta connection of the load. In delta-connected equipment, the delta is usually formed close to the windings and line wires only are brought out.

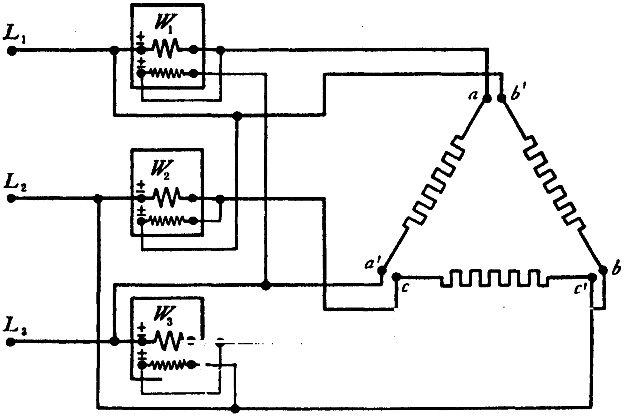


Fig. 4-25. Three Single-phase Wattmeters Connected to Measure Power in a Delta System.

Hence, the connection of Fig. 4-25 is not practicable for delta-connected equipment of this type. The power in 3-wire, 3-phase circuits, whether wye or delta-connected, may be measured by 3 single-phase wattmeters,

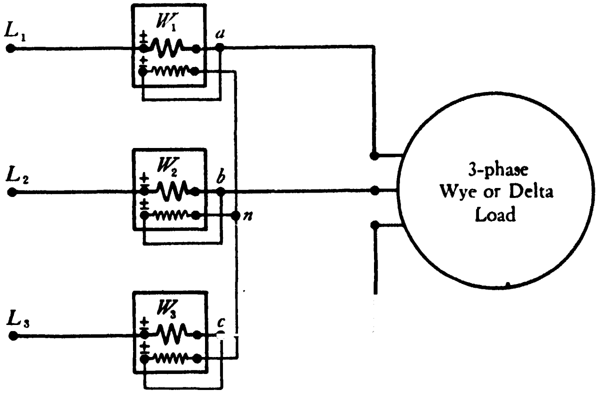


Fig. 4-26. Three Single-phase Wattmeters Connected to Measure Power Drawn by a 3-wire, 3-phase Load.

as shown in Fig. 4-26, and with this connection the difficulties mentioned above are eliminated.

It is not essential that the potential elements of the 3 wattmeters in Fig. 4-26 have equal impedances. In fact, the potential element impedance of one wattmeter can be short-circuited and the other two wattmeters will measure the total power of the circuit. This is equivalent to connecting the junction *n* in Fig. 4-26 to one of the line wires. If, for example, *n* is connected to *c*, wattmeter W_3 will be of no further value in making the measurement, and hence, W_3 can be removed. Fig. 4-27 shows 2 single-phase wattmeters connected to read the power in a 3-phase, 3-wire circuit. The 2-wattmeter method cannot be used in 3-phase, 4-wire circuits. Three wattmeters are required when 4 wires are used. In general, the minimum number of wattmeters required for measuring power in a polyphase system having *n* wires is $n - 1$.

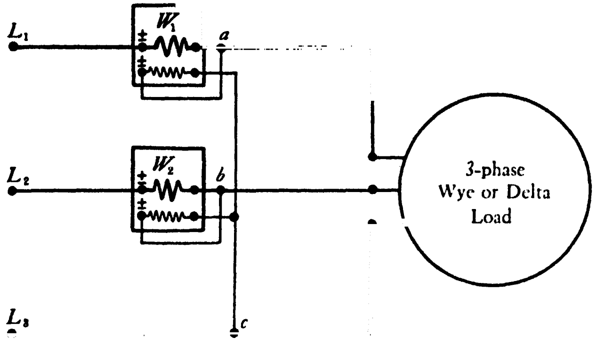


Fig. 4-27. Two Single-phase Wattmeters Connected to Measure Power Drawn by a 3-wire, 3-phase Load.

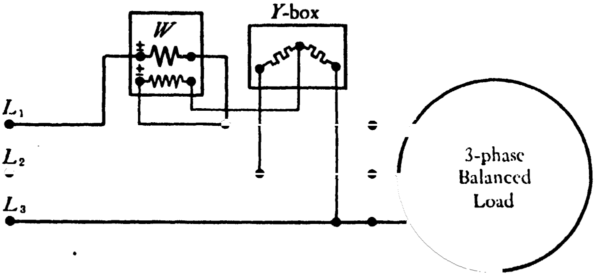


Fig. 4-28. Single-phase Wattmeter and Y-box Connected to Measure Power drawn by a Balanced 3-wire, 3-phase Load.

Under balanced conditions a *Y-box* may be substituted for two wattmeters in Fig. 4-26. The Y-box (Fig. 4-28) consists of two equal impedances, each equal to the impedance of the wattmeter potential element with which it is used. The total power is 3 times the power indicated by the wattmeter.

19. The 2-wattmeter Method. Assume the load of Fig. 4-29*a* to be a balanced delta-connected load in which the phase current lags the phase voltage by an angle θ. The voltage impressed on the potential element of W_1 when traced from the ± marking is E_{ab}, and the current in the current element of W_1 when traced from the ± marking is $I_{a'a}$. The wattage indicated by W_1 is

$$P_1 = E_{ab}I_{a'a} \cos \text{(the angle between } E_{ab} \text{ and } I_{a'a})$$

From the vector diagram (Fig. 4-29*b*) it is evident that the angle between E_{ab} and $I_{a'a}$ equals $(30 + \theta)$ time-degrees. Also, E_{ab} and $I_{a'a}$ are line values. Then,

$$P_1 = E_L I_L \cos (30 + \theta) \tag{4-40}$$

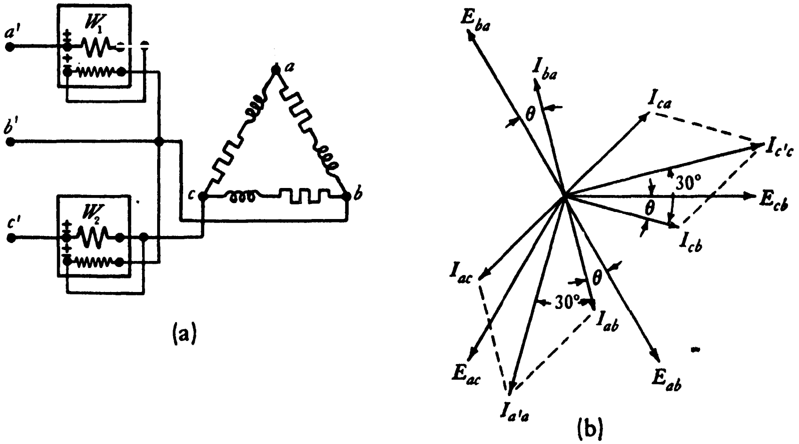


Fig. 4–29. Two Single-phase Wattmeters Connected to Measure Power Drawn by a Delta Load.

The voltage impressed on the potential element of W_2 when traced from the ± marking is E_{cb}, and the current in the current element of W_2 when traced from the ± marking of W_2 is $I_{c'c}$. The wattage indicated by W_2 is

$$P_2 = E_{cb}I_{c'c} \cos \text{(the angle between } E_{cb} \text{ and } I_{c'c})$$

From the vector diagram, it is evident that the angle between E_{cb} and $I_{c'c}$ equals $(30 - \theta)$ and

$$P_2 = E_L I_L \cos (30 - \theta) \tag{4-41}$$

Expanding and simplifying equations 4-40 and 4-41,

$$\begin{aligned}
P_1 &= E_L I_L (\cos 30 \cos \theta - \sin 30 \sin \theta) \\
&= 0.866 E_L I_L \cos \theta - 0.5 E_L I_L \sin \theta \\
P_2 &= E_L I_L (\cos 30 \cos \theta + \sin 30 \sin \theta) \\
&= 0.866 E_L I_L \cos \theta + 0.5 E_L I_L \sin \theta \\
P_1 + P_2 &= 1.732 E_L I_L \cos \theta = \sqrt{3}\, E_L I_L \cos \theta
\end{aligned}$$

From equation 4-29 the total power,

$$P_T = \sqrt{3}\, E_L I_L \cos\theta$$

Hence,

$$P_T = P_1 + P_2$$

Attention is called to the importance of current and voltage directions in the wattmeter elements. The ± terminal of the potential element of each wattmeter must be connected to the line wire in which the current coil of that particular wattmeter is inserted.

When power in a balanced load is measured by the two-wattmeter method, the indications of the two instruments will not be the same except at unity power factor. At unity power factor the phase angle is zero degrees, and

$$\begin{aligned} P_1 &= E_L I_L \cos(30^\circ + \theta^\circ) = 0.866 E_L I_L \\ P_2 &= E_L I_L \cos(30^\circ - \theta^\circ) = 0.866 E_L I_L \end{aligned}$$

Hence, P_1 and P_2 each read one-half the total power.

When the power factor is 50 per cent, the phase angle is 60°, and

$$\begin{aligned} P_1 &= E_L I_L \cos(30^\circ + 60^\circ) = 0 \\ P_2 &= E_L I_L \cos(30^\circ - 60^\circ) = 0.866 E_L I_L \end{aligned}$$

Under this condition P_2 reads the total power in the circuit.

When the power factor of the circuit is less than 50 per cent, the phase angle is greater than 60° and the angle $(30 + \theta)$ is greater than 90°. Thus, $\cos(30 + \theta)$ will be negative and the pointer of W_1 will reverse. To obtain the total power in this case, the current element of W_1 should be reversed and the reading of W_1 should be subtracted from that of W_2.

It can be shown by use of the calculus that the 2 wattmeters (Fig. 4-27) measure the total power in an unbalanced, as well as in a balanced, 3-wire, 3-phase system. Example 4-9 verifies this proof for a specific case.

Example 4-9: Show that power in the unbalanced delta-connected circuit of example 4-6 can be measured by the 2-wattmeter method.

Solution. Phase sequence E_{cb}, E_{ac}, E_{ba}. The total power of the load is first determined by finding the power in each phase and summarizing. Then the power read by the 2 wattmeters, when connected as shown in Fig. 4-27, is determined and compared with the first calculation. All values of power are computed, using voltages and currents expressed in rectangular coordinates. That is,

$$\begin{aligned} P &= E_H I_H + E_V I_V \\ P_{cb} &= (220)(14.76) + (0)(10.33) = 3245 \text{ watts} \\ P_{ac} &= (-110)(-13.02) + (-190.5)(-11.53) = 3627 \text{ watts} \\ P_{ba} &= (-110)(2.63) + (190.5)(21.84) = 3871 \text{ watts} \\ P_T &= P_{cb} + P_{ac} + P_{ba} = 10{,}743 \text{ watts} \end{aligned}$$

With the 2-wattmeter method, as shown in Fig. 4-27, the current in $W_1 = I_{a'a}$ and the potential applied to $W_1 = E_{ac}$. Then

$$P_1 = (-110)(-15.65) + (-190.5)(-33.37) = 8073 \text{ watts}$$

The current in $W_2 = I_{b'b}$ and the potential applied to $W_2 = E_{bc}$. Then

$$P_2 = (-220)(-12.13) + (0)(11.55) = 2670 \text{ watts}$$

$$P_1 + P_2 = 10{,}743 \text{ watts}$$

Hence,

$$P_1 + P_2 = P_T$$

20. Determining Power Factor from the 2-wattmeter Method. Equation 4-40 divided by equation 4-41 gives,

$$\frac{P_1}{P_2} = \frac{E_L I_L \cos(30 + \theta)}{E_L I_L \cos(30 - \theta)} = \frac{\cos(30 + \theta)}{\cos(30 - \theta)}$$

Expanding and cross multiplying, we have

$$P_1 \cos 30 \cos \theta + P_1 \sin 30 \sin \theta = P_2 \cos 30 \cos \theta - P_2 \sin 30 \sin \theta$$

$$0.866\, P_1 \cos \theta + 0.5\, P_1 \sin \theta = 0.866\, P_2 \cos \theta - 0.5\, P_2 \sin \theta$$

Dividing by 0.5 cos θ to simplify, we have

$$1.732P_1 + P_1 \tan \theta = 1.732P_2 - P_2 \tan \theta$$

$$\tan \theta = 1.732\left(\frac{P_2 - P_1}{P_2 + P_1}\right) \tag{4-42}$$

The phase angle θ may be determined by equation 4-42 when the 2-wattmeter method is used to measure power in a balanced 3-phase system. Attention is called to the fact that P_2 in equation 4-42 represents the larger of the 2 wattmeter readings when the power factor is other than unity.

Since the power factor equals the cosine of the phase angle, the power factor may also be determined from the ratio of the 2 wattmeter readings.

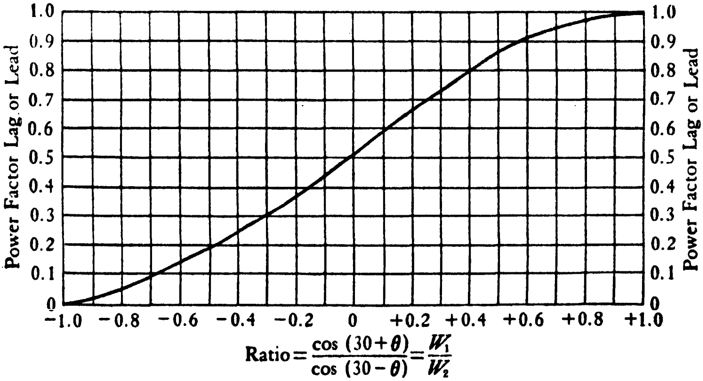


Fig. 4-30. Power Factor of a Balanced 3-phase Load from Ratio of W_1/W_2.

The graph (Fig. 4-30) shows the relationship between the power factor and the ratio of W_1/W_2.

Example 4-10: Two wattmeters are connected to measure the power required by a balanced 3-phase load. W_1 indicates 2400 watts and W_2, 8400 watts. Determine the power factor of the load.

Solution.

$$\tan\theta = 1.732\left(\frac{8400 - 2400}{8400 + 2400}\right) = 0.554$$

$$\theta = 28^\circ\,59'$$

$$\cos\theta = 0.8746 \qquad \textit{Ans.}$$

Problem 4-20: Prove that 2 wattmeters, when properly connected, read the power in a balanced wye load. Assume the load to have a lagging phase angle θ, and to be connected across a 3-phase, 3-wire system. A circuit and a vector diagram should accompany the proof.

Problem 4-21: Show that power in the unbalanced load of problem 4-14 can be measured by the 2-wattmeter method.

Problem 4-22: Show that power in the unbalanced load of problem 4-15 can be measured by the 2-wattmeter method.

Problem 4-23: W_1 indicates $-$ 150 watts and $W_2 +$ 840 watts when connected to measure the power required by a balanced 3-phase load. Determine the power factor of the load and check results with the graph (Fig. 4-30).

21. Voltage and Current Relations in Balanced 2-phase Systems. The two coils $a'a$ and $b'b$ of the alternator (Fig. 4-31) are so spaced on the surface of the armature as to cause 90 time-degrees displacement between their emfs. The 2 single-phase emfs generated by this alternator form a *2-phase* system. Fig. 4-32 shows the emf waves and vectors of a balanced 2-phase system. When the 2 separate phases of the system are electrically insulated from one another, and power is transmitted over 4 wires, the 2-phase system is treated as 2 independent single-phase systems.

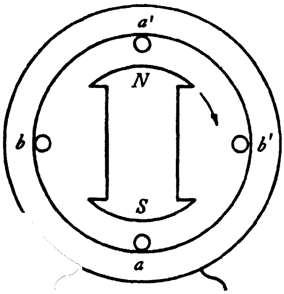


Fig. 4-31. Elementary 2-Phase Alternator.

A *3-wire, 2-phase* system can be formed by connecting an end of coil $a'a$ to an end of coil $b'b$. In the schematic diagram of a 2-phase alternator (Fig. 4-33), coil end a' is connected to coil end b' and line connections are made at a, b, and the junction $a'b'$. The wire connected to the junction of the 2 phases is known as the *neutral* wire and those connected to the other ends of the coils are usually referred to as *line* wires. The emf between the 2 line wires is the sum of the phase emfs combined subtractively at 90 time-degrees. That is,

$$E_{ba} = E_{bb'} + E_{a'a}$$

E_{ba} represents the emf between line wires, and $E_{bb'}$ and $E_{a'a}$ are each phase voltages. Substituting the symbol E_L for E_{ba}, and E_P for each $E_{bb'}$ and $E_{a'a}$, the magnitude of E_L in terms of E_P is

$$E_L^2 = \sqrt{E_P^2 + E_P^2} = \sqrt{2E_P^2}$$

$$E_L = \sqrt{2}\,E_P = 1.41E_P \tag{4-43}$$

It is evident from Fig. 4-33 that in a balanced 3-wire, 2-phase system E_L leads or lags one of its component phase voltages by 45° and the other by 135°, depending on whether E_L is taken from b to a or from a to b.

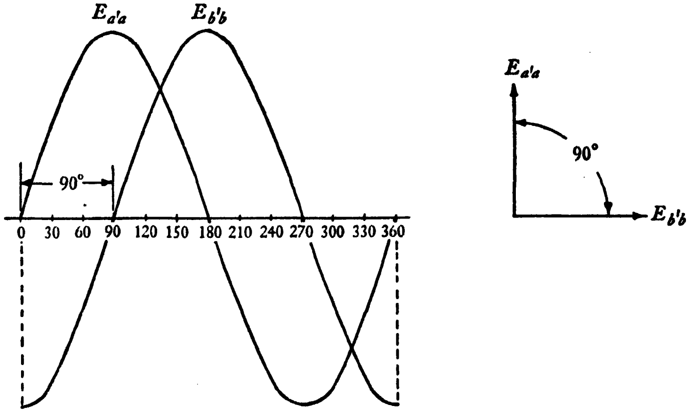


Fig. 4-32. Balanced 2-phase Emf Waves and Vectors.

A 2-phase load is connected to a 2-phase generator as shown in Fig. 4-34. Writing Kirchhoff's current equation for the junction n, and solving for the current in the neutral wire,

$$I_{nn'} = I_{a'a} + I_{b'b}$$

When the load is balanced $I_{a'a}$ and $I_{b'b}$ are equal in magnitude and are displaced 90 time-degrees from one another. If the current in the neutral

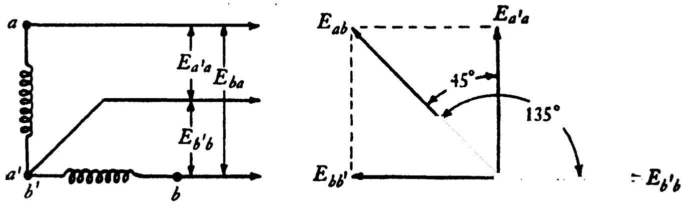


Fig. 4-33. Connection and Emf Vector Diagram for a 3-wire, 2-phase System.

wire is represented by I_n and the current in each phase or line wire is represented by I_P the magnitude of I_n in terms of I_P is

$$I_n = \sqrt{I_P^2 + I_P^2} = \sqrt{2I_P^2}$$

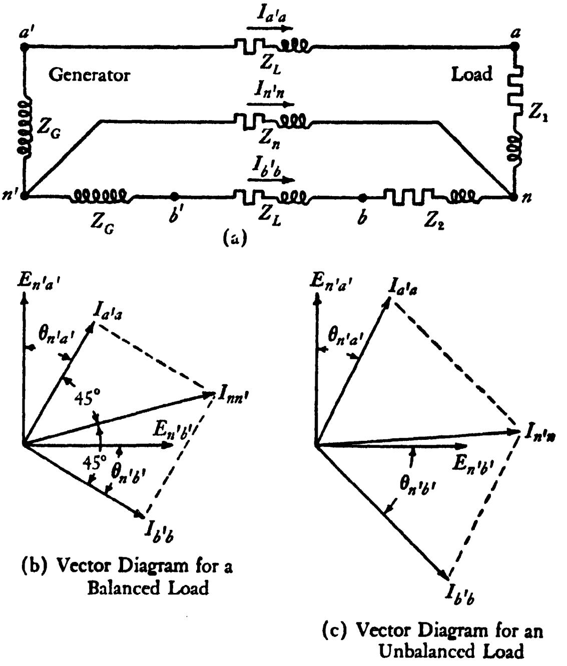
$$I_n = \sqrt{2}\, I_P = 1.41 I_P \qquad (4\text{-}44)$$

In a balanced 3-wire, 3-phase system I_n lags one of its phase current components and leads the other by 45 time-degrees.

Example 4-11: The phase voltage at the terminals of the load in a balanced 3-wire, 2-phase system is 240 volts. Determine the magnitude of the voltage between line wires.

Solution.

$$E_L = 1.41 V_P = 1.41 \times 240 = 338.4 \text{ volts} \qquad \textit{Ans.}$$



(b) Vector Diagram for a Balanced Load

(c) Vector Diagram for an Unbalanced Load

Fig. 4-34. Circuit and Vector Diagrams for a 3-wire, 2-phase System.

22. Unbalanced 2-phase Systems. Unbalanced 2-phase systems are solved in a similar manner to that used for unbalanced 3-wire, 3-phase systems. Assume the load of Fig. 4-34 to be unbalanced. Kirchhoff's current equation at junction *n* yields

$$I_{a'a} + I_{b'b} + I_{n'n} = 0 \qquad (4\text{-}45)$$

Assuming one positive circuit direction for tracing to be $n'a'ann'$,

$$E_{n'a'} - I_{a'a}Z_G - I_{a'a}Z_L - I_{a'a}Z_1 - I_{nn'}Z_n = 0 \qquad (4\text{-}46)$$

And assuming a second positive circuit direction for tracing to be $n'b'bnn'$,

$$E_{n'b'} - I_{b'b}Z_G - I_{b'b}Z_L - I_{b'b}Z_2 - I_{nn'}Z_n = 0 \qquad (4\text{-}47)$$

Substituting the value of $I_{nn'}$ from equation 4-45 into equations 4-46 and 4-47 and grouping terms, we have

$$E_{n'a'} - I_{a'a}(Z_G + Z_L + Z_1 + Z_n) - I_{b'b}Z_n = 0 \qquad (4\text{-}48)$$
$$E_{n'b'} - I_{b'b}(Z_G + Z_L + Z_2 + Z_n) - I_{a'a}Z_n = 0 \qquad (4\text{-}49)$$

To simplify equations 4-48 and 4-49, set

$$Z_a = Z_G + Z_L + Z_1 + Z_n$$
$$Z_b = Z_G + Z_L + Z_2 + Z_n$$

Substituting the symbols Z_a and Z_b into equations 4-48 and 4-49 and solving simultaneously for $I_{a'a}$,

$$I_{b'b} = \frac{E_{n'a'} - I_{a'a}Z_a}{Z_n}$$
$$I_{b'b} = \frac{E_{n'b'} - I_{a'a}Z_n}{Z_b}$$

By comparison,

$$\frac{E_{n'a'} - I_{a'a}Z_a}{Z_n} = \frac{E_{n'b'} - I_{a'a}Z_n}{Z_b}$$
$$I_{a'a} = \frac{E_{n'a'}Z_{b'} - E_{n'b'}Z_n}{Z_aZ_b - Z_n^2} \qquad (4\text{-}50)$$

It can be shown in a similar manner that

$$I_{b'b} = \frac{E_{n'b'}Z_a - E_{n'a'}Z_n}{Z_aZ_b - Z_n^2} \qquad (4\text{-}51)$$

After $I_{a'a}$ and $I_{b'b}$ have been determined by equations 4-50 and 4-51, $I_{nn'}$ can be determined by substituting in equation 4-45.

23. Unbalanced 2-phase Systems — Fixed Voltages Assumed at the Load Terminals. When the generator, line wire, and neutral wire impedances in an unbalanced 3-wire, 2-phase system are small enough in comparison to the load impedances to be neglected, the voltage at the terminals of the load may be considered as fixed and the computation of currents in terms of voltages and impedances is greatly simplified. Example 4-12 illustrates the method of solving unbalanced 3-wire, 2-phase systems when the phase voltages at the load terminals are assumed to be fixed.

Example 4-12: In the unbalanced 3-wire, 2-phase circuit (Fig. 4-35), $Z_1 = 9 + j\,2$ ohms, and $Z_2 = 8 + j\,5$ ohms. $E_{an} = 0 + j\,440$ volts, and $E_{bn} = 440 + j\,0$ volts. Determine the currents in the line and neutral wires.

Solution. See Fig. 4-35.

$$
\begin{aligned}
I_{an} &= \frac{E_{an}}{Z_1} = \frac{0 + j\,440}{9 + j\,2} \\
&= 10.4 + j\,46.6 = 57.8\underline{/77^\circ\ 28'} && \textit{Ans.} \\
I_{bn} &= \frac{E_{bn}}{Z_2} = \frac{440 + j\,0}{8 + j\,5} \\
&= 39.6 - j\,24.8 = 46.8\overline{\backslash 32^\circ}\ \text{amp} && \textit{Ans.} \\
I_{nn'} &= I_{an} + I_{bn} \\
&= 50.0 + j\,21.8 = 54.6\underline{/23^\circ\ 34'}\ \text{amp} && \textit{Ans.}
\end{aligned}
$$

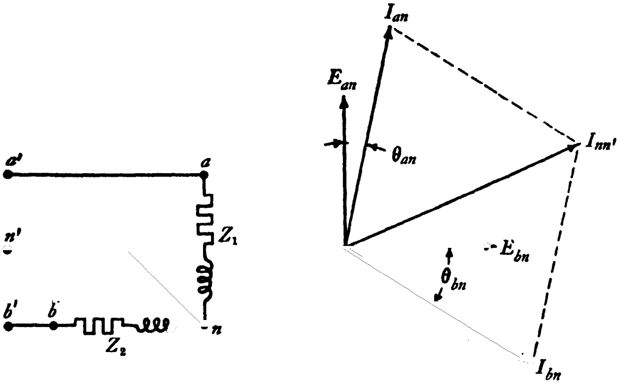


Fig. 4-35. Diagrams for Example 4-12.

Problem 4-24: The phase current in a balanced, 3-wire, 2-phase system has a magnitude of 415 amp, and the phase voltage at the load terminals has a magnitude of 220 volts. Determine: (*a*) the magnitude of the current in the neutral wire; (*b*) the magnitude of the voltage between line wires.

Problem 4-25: Given the following values for the circuit of Fig. 4-34, determine: (*a*) the current in each line wire and the neutral wire; (*b*) the phase voltages across the terminals of the load; (*c*) the line impedance voltages; (*d*) the neutral impedance voltage; (*e*) the generator impedance voltages. Draw a vector diagram to scale showing the currents and voltages computed in parts (*a*) to (*e*) inclusive.

$$
\begin{aligned}
E_{n'a'} &= 460\underline{/90^\circ} = 0 + j\,460\ \text{volts} \\
E_{n'b'} &= 460\underline{/0^\circ} = 460 + j\,0\ \text{volts} \\
Z_G &= 0.2 + j\,0.6\ \text{ohms} \\
Z_L &= 1.0 + j\,0.8\ \text{ohms} \\
Z_n &= 0.6 + j\,0.5\ \text{ohms} \\
Z_1 &= 20 + j\,10\ \text{ohms} \\
Z_2 &= 15 + j\,15\ \text{ohms}
\end{aligned}
$$

Problem 4-26: An impedance $Z_1 = 1.2 + j\,0.4$ ohms is connected across one of the phase voltages of a 2-phase system $E_{an} = 0 + j\,220$ volts. A second impedance, $Z_2 = 0.9 + j\,0.5$ ohms, is connected across the second phase voltage of the system $E_{bn} = 220 + j\,0$ volts. Determine the currents in the line and neutral wires.

24. Quarter-Phase Systems. If the centers of the generator or transformer phase windings of a 2-phase system are interconnected as shown in Fig. 4-36, the system formed is called a *quarter-phase* (*4-phase*) system. Four line wires are required for balanced quarter-phase systems, and, in addition, a neutral wire should be used if the system is subjected to unbalanced conditions. The quarter-phase system is seldom used because of the number of transmission wires required and because a system of this type is easily unbalanced.

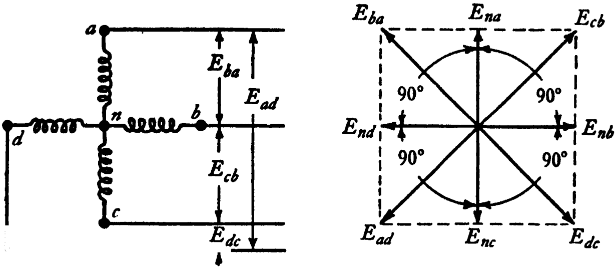


Fig. 4-36. Connection and Vector Diagrams of Emfs of a Quarter-Phase Alternator.

25. Power in 2-Phase and Quarter-Phase Systems. In a 2-phase system, whether 3-wire or 4-wire, the total power equals the sum of the power in the two phases.

$$P_T = P_1 + P_2 \tag{4-52}$$

Two-phase power can be measured with 2 single-phase wattmeters or with one 2-element wattmeter. Fig. 4-37 shows the connection for measuring power in a 3-wire, 2-phase system with 2 single-phase wattmeters.

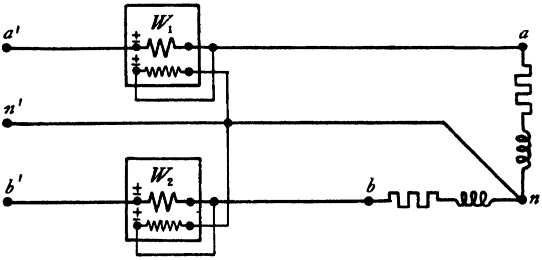


Fig. 4-37. Two Single-Phase Wattmeters Connected to Measure Power Drawn by a 3-Wire, 2-Phase Load.

The total power in a quarter-phase system equals the sum of the power in the 4 phases. Three single-phase wattmeters are used to measure power in the 4-wire, quarter-phase system shown in Fig. 4-38. When a neutral or fifth wire is used with the quarter-phase system, 4 single-phase wattmeters, one for each phase, are required.

26. Six-phase Systems. Six emfs of equal magnitude displaced from one another so that an angle of 60 time-degrees exists between each 2 consecutive phase emfs form a *6-phase* system. A 6-phase emf may be generated directly by a 6-phase alternator, but in practice 6-phase systems are usually obtained by transformation from 3-phase systems. Synchronous converters and rectifiers are frequently operated 6-phase from 3-phase systems by connecting the converter or rectifier to the 3-phase system through transformers. Since the application of 6-phase is practically limited to converters and rectifiers, further discussion of this system will be omitted here.

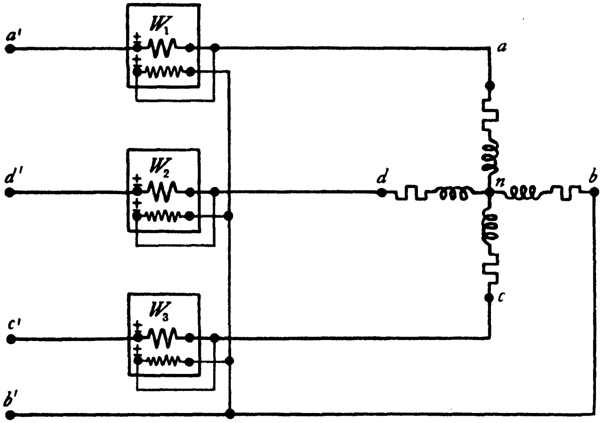


Fig. 4-38. Three Single-Phase Wattmeters Connected to Measure Power Drawn by a 4-Wire, Quarter-Phase Load.

27. Volt-amperes, Watts, and Vars in Polyphase Systems. The method of solving single-phase circuit problems involving volt-amperes, watts, and vars as discussed in article 14, Chapter III, is also applicable to balanced polyphase circuits. Equation 3-45 holds for balanced polyphase circuits as well as for single-phase circuits.

The total kva, kw, and kvar for unbalanced polyphase systems have little value, since a definite ratio between single-phase and polyphase quantities does not exist under unbalanced conditions. Hence, when unbalanced polyphase circuits involving kva, kw, and kvar are to be solved, they should be considered as a group of single-phase systems and each single-phase system should be treated independently of the other. In unbalanced polyphase systems there are likely to be as many different phase angles as there are phases in the system, and since power factor is the cosine of the phase angle, each phase of an unbalanced polyphase system may have a different power factor. For this reason the terms *phase angle* and *power* *factor*, when used in reference to polyphase systems, have no meaning when the system is unbalanced.

Example 4-13: A 3-wire, 3-phase transmission line supplies the following balanced, 3-phase loads: 275 kw at 80 per cent lagging power factor; 350 kw at 90 per cent lagging power factor; and 125 kw at 90 per cent leading power factor. Determine: (*a*) the total kw; (*b*) the total kva; (*c*) the total kvar; (*d*) the power factor of the combined loads.

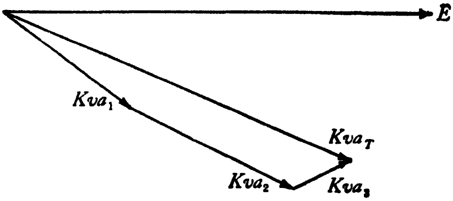


Fig. 4-39. Diagram for Example 4-13.

Solution. See Fig. 4-39.

$$(a)\qquad P_T = P_1 + P_2 + P_3$$
$$= 275 + 350 + 125 = 750 \text{ kw} \qquad \textit{Ans.}$$

$$(b)\quad \text{Kva}_1\overline{\backslash\theta_1} = \frac{\text{Kw}_1}{0.8}\overline{\backslash\cos^{-1}0.8}$$
$$= 344\overline{\backslash 36^\circ 52'} = 275 - j\,206.4 \text{ kva}$$
$$\text{Kva}_2\overline{\backslash\theta_2} = \frac{\text{Kw}_2}{0.9} = \overline{\backslash\cos^{-1}0.9}$$
$$= 389\overline{\backslash 25^\circ 50'} = 350 - j\,169.5 \text{ kva}$$
$$\text{Kva}_3/\underline{\theta_3} = \frac{\text{Kw}_3}{0.9}/\underline{\cos^{-1}0.9}$$
$$= 139/\underline{25^\circ 50'} = 125 + j\,60.7 \text{ kva}$$
$$\text{Kva}_T\overline{\backslash\theta_T} = 275 - j\,206.4 + 350 - j\,169.5 + 125 + j\,60.7$$
$$= 750 - j\,315.2 = 814\overline{\backslash 22^\circ 47'} \text{ kva} \qquad \textit{Ans.}$$

(*c*) The total kilovolt-amperes-reactive equals the total kva times the sine of the phase angle for the entire circuit. This is the second term of the complex expression for kva_T. Hence,

$$\text{Kvar}_T = 315.2 \text{ kvar} \qquad \textit{Ans.}$$

$$(d)\qquad PF_T = \cos\theta_T = \cos 22^\circ 47' = 0.92 \text{ lag} \qquad \textit{Ans.}$$

Problem 4-27: A 3-conductor cable is carrying a balanced 3-phase load of 840 kva at a 60 per cent lagging power factor. It is desired to connect a second load of 600 kva which will have a lagging power factor of 80 per cent. If a 300-kva synchronous condenser which operates at 10 per cent leading power factor is also connected to the cable (*a*) what will be the total kva supplied by the cable? (*b*) what will be the power factor of the combined load on the cable?

Problem 4-28: A 3-phase feeder supplies the following balanced loads: 100-kva induction motor load at 80 per cent lagging power factor; a 3-phase resistance oven load of 24 kw; a 10-kva, 3-phase static capacitor; and a 15-kva, 3-phase electric furnace with a 50 per cent lagging power factor. Determine: (*a*) the total kva on the feeder; (*b*) the power supplied by the feeder; (*c*) the power factor of the feeder.

QUESTIONS

4-1. What are the advantages of a polyphase system when compared to a single-phase system?

4-2. Describe the difference between a balanced 2-phase system and a balanced 3-phase system.

4-3. Why are double-subscript notations used in denoting currents and voltages in polyphase systems?

4-4. What is meant by phase sequence and why is it necessary to state the phase sequence in solving unbalanced polyphase circuit problems?

4-5. How does a wye-connected, 3-phase system differ from a delta-connected, 3-phase system?

4-6. What relationship in respect to magnitude and angular position does the line voltage of a balanced wye-connected system have to its component phase voltages?

4-7. Give the relationship between the line and phase currents in a balanced wye-connected system.

4-8. What relationship in respect to magnitude and angular position does the line current in a balanced delta-connected system have to its component phase currents?

4-9. Give the relationship between the current in the line wires and the current in the neutral wire for an unbalanced, 4-wire, wye-connected system.

4-10. What procedure is recommended in determining currents in an unbalanced, 4-wire wye-connected system, the emfs and impedances being known?

4-11. What procedure is recommended in determining currents in an unbalanced delta-connected system, the emfs and impedances being known?

4-12. Under what conditions can fixed voltages be assumed at the terminals of polyphase loads without too great an error in the solution?

4-13. State the minimum number of single-phase wattmeters required for measurement of power: (*a*) in a 3-wire, 3-phase system; (*b*) in a 4-wire, 3-phase system; (*c*) in a 3-wire, 2-phase system; (*d*) in a 5-wire quarter-phase system.

4-14. When the 2-wattmeter method is used to measure power in a 3-phase system under what condition: (*a*) do the 2 wattmeters give the same readings; (*b*) does one wattmeter read twice as much as the other; (*c*) does one wattmeter reverse?

4-15. What relationship in respect to magnitude and angular position does the voltage between line wires of a balanced, 3-wire, 2-phase system have to its component phase voltages?

4-16. What relationship in respect to magnitude and angular position does the current in the neutral wire of a balanced, 3-wire, 2-phase system have to the currents in the line wires?

4-17. The primary windings of 3 single-phase transformers are connected to a 3-phase system. The voltage across each transformer secondary is 230 volts and the secondary coils are being connected delta. Two of the junctions have been made, but a voltage check for zero voltage before closing the delta shows 460 volts instead of zero. What is wrong with the connection and how can it be corrected?

4-18. A distribution panel is fed by 3 wires. How would you determine whether the system is: (*a*) 3-wire direct-current; (*b*) 3-wire single-phase; (*c*) 3-wire, 2-phase; (*d*) 3-wire, 3-phase?

PROBLEMS

4-29. A 100-hp (output), 3-phase, 220-volt induction motor operates at an efficiency of 88 per cent and a power factor of 92 per cent when fully loaded. What is the rated current per terminal when carrying full load?

4-30. A 3-phase induction motor draws 7864 watts and the line voltage across its terminals is 220 volts. A line ammeter reads 24 amperes. (*a*) What is the power factor of the motor? (*b*) If the total power is read by 2 wattmeters, what is the reading of each instrument?

4-31. Both ends of the phase coils of a 5000-kva synchronous condenser are brought out to provide for wye or delta connection. The stator winding is designed for a phase voltage of 6600 volts. The 5000-kva rating is an input rating. Determine: (*a*) the line voltage and line current rating when the windings are wye-connected; (*b*) when delta-connected.

4-32. A 3-phase, 4-wire feeder supplies energy to a 300-kva, 85 per cent power factor, balanced wye-connected load. It is desired to add a single-phase load of 125 kva at unity power factor which is to be connected from line *a* to neutral. Assuming the line voltage at the terminal of the load to be fixed at 4000 volts, determine: (*a*) the voltage across the additional single-phase load; (*b*) the current in each line wire and the neutral wire after the single-phase load has been added.

4-33. Sixteen resistor units, each having a rating of 7.6 kw at 220 volts, are connected to a 3-wire, 3-phase source, as illustrated in Fig. 4-40. Determine the current in each line wire when the voltage between line wires is assumed to be fixed at 220 volts.

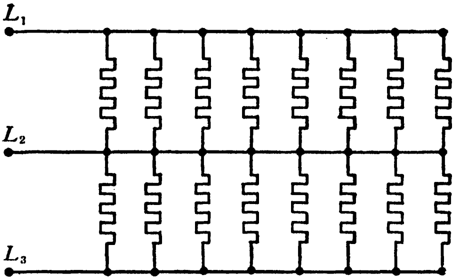


Fig. 4-40. Circuit for Problem 4-33.

4-34. A 3-phase, 4-wire system supplies power to: a 40-kva load at 80 per cent lagging power factor which is connected from line *a* to neutral; a 36-kva load at 90 per cent leading power factor which is connected from line *b* to neutral; and a 50-kva load at 90 per cent lagging power factor which is connected from line *c* to neutral. The phase voltage of the system is assumed to be fixed at 318 volts. Determine: (*a*) the current in each line wire and the neutral; (*b*) the total power taken by the load.

4-35. Three impedances each equal to $6.4 + j\,3.8$ ohms are connected to form a delta load. When the line voltage at the terminals of the load is assumed to be fixed at 2300 volts, determine: (*a*) the phase current; (*b*) the line current; (*c*) the wattage required by the load.

4-36. A test reading taken on a 100-hp, 3-phase induction motor shows 440 volts between line wires, 110 amp, and 77 kw input. The wattage input is measured by the 2-wattmeter method. Determine the wattage indicated by each instrument.

4-37. Two wattmeters used to read the power taken by a balanced 3-phase load indicate 628 and 482 watts respectively. What is the power factor of the load?

4-38. A 3-wire, 2-phase system supplies a 160-kva, 80 per cent lagging power factor load which is connected from line *a* to neutral; and a 90-kva, 90 per cent leading power factor load which is connected from line *b* to neutral. A unity power factor load of 70 kw is connected from line *a* to line *b* with no neutral connection. The phase voltage is assumed to be fixed at 2200 volts. Determine the current in each line wire and the neutral wire of the system.

4-39. The power taken by a 2-phase load is read by 2 wattmeters. Each instrument reads 728 watts. What is the power taken by the load?

4-40. What kva 3-phase static capacitor load is necessary to correct 150 kva of induction motor load from 75 per cent lagging power factor to 90 per cent lagging power factor?

4-41. A 3-phase, 3-wire system supplies 1500 kva at a 72 per cent lagging power factor and a 100-kva, 3-phase unity power factor load. If a 200-kva synchronous motor load at 80 per cent leading power factor is added, what will be: (*a*) the total kva; (*b*) the system power factor; (*c*) the total kw?

LABORATORY EXPERIMENTS

EXPERIMENT 4-1

Purpose. To measure currents, voltages, and power in balanced and unbalanced 3-phase, wye-connected systems.

Procedure. **1.** Connect 3 resistances having about the same values to form a wye load across a 4-wire, 3-phase source. Each resistance should be of ample rating so that it will not overheat with 150 per cent phase voltage applied to its terminals. If a 120/208-volt source is used, three 120-volt lampbanks, each with a slide-wire rheostat in series will make a very satisfactory wye load for this experiment. Insert an ammeter in each line wire and a fourth ammeter in the neutral wire. Connect 3 single-phase wattmeters to measure the power taken by the load, and connect voltmeters to measure line and phase voltages. Adjust the rheostats until the line ammeters read alike, i.e., balance the load. Under this condition the 3 wattmeters should also read alike and the ammeter in the neutral wire should read zero. Read the instruments and record data.

2. Disconnect the neutral formed by the potential elements of the 3 wattmeters from the neutral of the system. Read the instruments and record data.

3. With the wye load still balanced disconnect the neutral wire from the source. Read the instruments and record data.

4. Reconnect the neutral wire between the source and load. Also, reconnect the neutral formed by the potential elements of the wattmeters to the neutral of the load. Unbalance the wye-connected load so that the current and wattage of each phase differs. Read the instruments and record data.

5. Repeat part 2 with the unbalanced load of part 4.

6. Repeat part 3 with the unbalanced load of part 4.

7. Without disturbing the adjustment of the rheostats as set in part 6 interchange 2 line wires at the load terminals. Read the instruments and record data. Reconnect the 2 line wires in their original positions.

8. Disconnect the 3 single-phase wattmeters and connect 2 single-phase wattmeters with correct ratings to measure the power of the 3-wire, 3-phase system by the 2-wattmeter method. The potential elements of the 2 wattmeters will now be connected across line voltages, and, hence, they should have adequate voltage ratings. Make certain that the neutral wire to the source is disconnected and that the phase currents are identical with those obtained in part 6. Read the instruments and record data.

9. With the wattmeters connected to read power by the 2-wattmeter method, adjust the rheostats to give the same balanced phase current readings as obtained in part 3. Read the instruments and record data.

Report. 1. Submit a complete report in which the experimental data obtained during this experiment are used to substantiate the following: (*a*) in a balanced wye-connected system the line voltage equals $\sqrt{3}$ times the phase voltage in magnitude; (*b*) the current in the neutral of a wye system is the vector sum of the currents in the line; (*c*) the currents, voltages, and wattage in a balanced wye system is not changed by disconnecting the neutral wire; (*d*) the currents, phase voltages, and wattage vary when the neutral wire is disconnected if the wye load is unbalanced; (*e*) changing the phase sequence of the line voltages in an unbalanced wye circuit gives a different set of values; (*f*) the wattage of a 3-wire, wye system, whether balanced or unbalanced, can be read by the 2-wattmeter method.

2. Submit a vector diagram drawn to a convenient scale with the data of part 4 which shows that the current in the neutral wire equals the vector sum of the line currents.

EXPERIMENT 4-2

Purpose. To measure currents, voltages, and power in balanced and unbalanced 3-phase delta-connected systems.

Procedure. 1. Connect 3 resistances having about the same values to form a delta load across a 3-phase source. Each resistance should have ample rating so that it will not overheat when rated line voltage is applied to its terminals. If a 208- or 220-volt, 3-phase source is used, three 240-volt lampbanks, each with a slide-wire rheostat in series, will make a very satisfactory delta load for this experiment. Connect an ammeter and a single-phase wattmeter in each phase and connect ammeters in the line wires. Also, connect a voltmeter to measure the line voltage. Adjust the rheostats until the three ammeters in the phases read alike, i.e., balance the load. Read the instruments and record data.

2. Unbalance the delta system so as to obtain a different load in each phase. Read the instruments and record data.

3. Interchange 2-line wires at the load terminals to reverse the phase sequence without changing the circuit in any other way. Read the instruments and record data. Reconnect the line wires in their original position.

4. Disconnect and remove the 3 wattmeters and connect 2 single-phase wattmeters to measure the power of the unbalanced delta load of part 2 by the 2-wattmeter method. The wattmeters used for the 2-wattmeter method must have current elements with ratings of at least the highest value of line current. Adjust the rheostats if necessary so that the phase ammeters read the same as in part 2. Read all instruments and record data.

5. With the circuit connection of part 4, adjust the rheostats to give the same balanced phase current as used in part 1. Read the instruments and record data.

Report. 1. Submit a complete report in which the experimental data obtained during this experiment are used to substantiate the following: (*a*) in a balanced delta-connected system the magnitude of the line current equals $\sqrt{3}$ times the magnitude of the phase current; (*b*) in an unbalanced delta-connected system each line current equals the vector sum of its two component phase currents; (*c*) changing the phase sequence of the line voltages gives a different set of values when the delta circuit is unbalanced; (*d*) power in either a balanced or unbalanced delta-connected system can be measured by the 2-wattmeter method.

2. Draw a vector diagram to a convenient scale showing the voltages and currents in the unbalanced delta-connected system of part 2 and part 3.

EXPERIMENT 4-3

Purpose. To compute power factor of a balanced 3-phase load from the ratio of the 2 wattmeter readings when power is measured by the 2-wattmeter method, and to check the values of power factor obtained with computations from E, I, and W.

Procedure. Connect 2 wattmeters to read the power input to a 3-phase induction motor. Insert an ammeter in each line and connect a voltmeter to read line voltage. Take care to close the ammeter and wattmeter short-circuiting switches before starting the motor. Load the motor with a d-c generator and take readings of the instruments at several different loads.

Report. 1. Submit a complete report.

2. The data sheet should contain, in addition to the instrument readings, the power factor as calculated from readings of E, I, and W, and the power factor as calculated from the readings of W_1 and W_2.

EXPERIMENT 4-4

Purpose. To measure currents, voltages, and power in a 3-wire, 2-phase system.

Procedure. 1. Connect 2 resistances having about the same values to form a 2-phase load across a 2-phase source. The 2-phase source may be obtained from a 3-phase source by transformation. Connections of the transformers will be made by the laboratory instructor or under the close supervision of the instructor. If the 2-phase voltage does not exceed 240 volts from line to neutral a 240-volt lampbank in series with a slide-wire rheostat will make a satisfactory load for each phase. Connect wattmeters to read the power of each phase, ammeters to read the current in each line wire and the neutral wire, and voltmeters to read phase voltage and voltage across the line wires. Adjust the rheostats until the load is balanced. Read the instruments and record data.

2. Unbalance the load. Read the instruments and record data.

3. Reduce the resistance of one phase and connect a reactor in series to give a lagging power factor. Reduce the resistance of the second phase and connect a capacitor in series to give a leading power factor. Adjust the rheostats to give values which will be indicated accurately by the instruments without exceeding their ratings. Read the instruments and record data.

4. Without disturbing the neutral connection or the load, interchange the line wire connections. Read the instruments and record data.

Report. 1. Submit a complete report in which the experimental data obtained during the experiment are used to substantiate the following: (*a*) in a balanced 2-phase system the magnitude of the voltage between line wires equals $\sqrt{2}$ times the magnitude of the phase voltage; (*b*) in a balanced 2-phase system the magnitude of the current in the neutral wire equals $\sqrt{2}$ times the magnitude of the phase current; (*c*) in an unbalanced 2-phase system the current in the neutral wire equals the vector sum of the phase currents; (*d*) changing the phase sequence of an unbalanced 2-phase system changes the neutral current.

2. Draw vector diagrams to a convenient scale for the voltages and currents in each part of the experiment.

APPENDIX A

LABORATORY REPORT WRITING

General. It is not the intent of this article to discuss in detail the various methods of writing engineering reports, but rather to suggest a form of report writing that has proved quite satisfactory for the laboratory experiments referred to in this text. If the report is typed, it should be double spaced on one side only of $8\frac{1}{2} \times 11$-inch unruled paper. The headings for the divisions of the report should be emphasized by underlines. If the report is written longhand, it should be in ink on one side only of standard ruled notebook paper which is slightly smaller than the $8\frac{1}{2} \times 11$-inch unruled paper. When written longhand, the headings for the major divisions of the report should be neatly lettered. The cover or title sheet for the report should contain: the experiment number, the title of the experiment, the date, the name of the tester and writer, and the names of those who assisted with the test. Orderly arrangement and neatness are essential characteristics of a good report. The following outline for report writing is suggested.

EXPERIMENT NO. 00

Title.

Purpose. Insert a statement of purpose or objectives of the experiment. This may be copied from the instruction sheet.

Apparatus. List the principal apparatus used in performing the experiment and give the ranges and numbers of all instruments. If a specific piece of equipment having an attached nameplate is tested, the nameplate data should be included in the report.

Wiring Diagram. The circuit diagram should be placed on a separate sheet of $8\frac{1}{2} \times 11$-inch unruled paper. It should be drawn with instruments, should contain a title, date, and signature, and should be inked with India ink. Correct symbols with necessary identifications should be lettered or typed on the diagram. When two or more test connections of a different nature are used in performing the experiment, a separate wiring diagram for each connection should be included.

Theory. A statement of the theory pertaining to the experiment and a discussion in the student's own words showing the relationship between the theory and the experiment should be inserted in the report under this heading.

Description. When the experiment consists of a test on a specific piece of equipment, the principles on which the equipment functions often may be given by describing the equipment. In such cases *Description* may be substituted for *Theory*.

Procedure. This should consist of a brief story of the method used in performing the experiment. If the procedure suggested by the laboratory instruction outline is followed exactly as given, the procedure may be taken from the instructions.

Sample Calculation. When calculated data are entered on the data sheet a sample of each type of computation should be entered under this heading.

Data. All data, read and calculated, should be arranged in rows and columns. Usually, one of the variables, called the *independent variable*, is controlled by the tester. The other variables, called *dependent variables*, are not controlled by the tester, but are permitted to vary in accordance with the change of the independent variable. The values of the variables should be arranged to form the columns of the data sheet with the values of the independent variable arranged consecutively in the first column to the left. All data, read and computed, for each specific value of the independent variable should be arranged to form a row. Then, the number of rows should equal the number of readings. The heading at the top of each column should contain: the name or symbol of the variable represented by the column; the number and range of the instrument used to read the values in the column (omitted when data in the column is computed); and the multiplying constant when a constant other than unity is to be applied to the values in the column.

Curves. A curve or graph is the means of visually expressing the relationship between two variables. It is customary to plot the independent variable as abscissa and the dependent variable as ordinate. Suggestions for plotting graphs are given in Appendix B.

Discussion of Results. This section of the report should consist of a summary of the results obtained from the test, written in the student's own words. When the results of the test check closely with the underlying theory of the experiment, this fact should be mentioned. When the results fail to check with the theory the student should seek an explanation and state it in this section of the report. Instructions intended to aid the student in writing this section of the report are placed at the end of the laboratory instruction sheets. The heading *Conclusions* is sometimes used in place of *Discussion of Results.*

APPENDIX B

INSTRUCTIONS FOR PLOTTING GRAPHS

1. Use standard $8\frac{1}{2} \times 11$-inch cross-section paper with 20 divisions per inch.

2. Determine the range of each variable, i.e., the difference between its highest and lowest value. Select scales for these ranges which will be easy to plot and read. Decide whether the graph will fit best on the sheet with the $8\frac{1}{2}$-inch or the 11-inch side horizontal. In either case, an allowance should be made for a 1-inch margin on the ruled section of the sheet at the bottom and left side of the sheet.

3. Having once decided which side of the sheet to place horizontally, draw the horizontal axis for the graph 1-inch above the bottom grid of the ruled section and the vertical axis 1-inch to the right of the left grid.

4. Start the lower end of the scales at or near the intersection of the axes and letter or type the values of the major divisions of the scales along the axes. Also, letter or type the title of each variable and the unit in which it is measured along the axis for that variable.

5. Plot the points and draw small circles around them. Draw a smooth curve through as many points as possible. Frequently, a smooth curve cannot be drawn through all of the points. Those points which do not lie on the curve should not be removed from the graph as they indicate the inaccuracy of the test.

6. Usually, when two or more curves having the same independent variable are to be drawn, they are placed on the same sheet of cross-section paper. In cases of this type, different scales may have to be used for the different dependent variables, and the margin between the vertical axis and the left grid may have to be increased.

7. A title, experiment number, and date should be lettered or typed on each curve sheet. When the sheet contains more than one curve, each curve should be labeled for identification. In addition, the draftsman should affix his signature to the curve sheet.

8. Curve sheets should be inked with India ink.

APPENDIX C

NATURAL TRIGONOMETRIC FUNCTIONS

Natural Sines and Cosines

′	0°		1°		2°		3°		4°		′
	N. sine	N. cos.	N. sine	N. cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	
0	.00000	1.00000	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	**60**
1	029	000	774	984	519	938	263	861	.07005	754	59
2	058	000	803	984	548	937	292	860	034	752	58
3	087	000	832	983	577	936	321	858	063	750	57
4	116	000	862	983	606	935	350	857	092	748	56
5	145	000	891	982	635	934	379	855	121	746	55
6	175	000	920	982	664	933	408	854	150	744	54
7	204	000	949	981	693	932	437	852	179	742	53
8	233	000	978	980	723	931	466	851	208	740	52
9	262	000	.02007	980	752	930	495	849	237	738	51
10	.00291	1.00000	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	**50**
11	320	.99999	065	979	810	927	553	846	295	734	49
12	349	999	094	978	839	926	582	844	324	731	48
13	378	999	123	977	868	925	611	842	353	729	47
14	407	999	152	977	897	924	640	841	382	727	46
15	436	999	181	976	926	923	669	839	411	725	45
16	465	999	211	976	955	922	698	838	440	723	44
17	495	999	240	975	984	921	727	836	469	721	43
18	524	999	269	974	.04013	919	756	834	498	719	42
19	553	998	298	974	042	918	785	833	527	716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	**40**
21	611	998	356	972	100	916	844	829	585	712	39
22	640	998	385	972	129	915	873	827	614	710	38
23	669	998	414	971	159	913	902	826	643	708	37
24	698	998	443	970	188	912	931	824	672	705	36
25	727	997	472	969	217	911	960	822	701	703	35
26	756	997	501	969	246	910	989	821	730	701	34
27	785	997	530	968	275	909	.06018	819	759	699	33
28	814	997	560	967	304	907	047	817	788	696	32
29	844	996	589	966	333	906	076	815	817	694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	**30**
31	902	996	647	965	391	904	134	812	875	689	29
32	931	996	676	964	420	902	163	810	904	687	28
33	960	995	705	963	449	901	192	808	933	685	27
34	989	995	734	963	478	900	221	806	962	683	26
35	.01018	995	763	962	507	898	250	804	991	680	25
36	047	995	792	961	536	897	279	803	.08020	678	24
37	076	994	821	960	565	896	308	801	049	676	23
38	105	994	850	959	594	894	337	799	078	673	22
39	134	994	879	959	623	893	366	797	107	671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	**20**
41	193	993	938	957	682	890	424	793	165	666	19
42	222	993	967	956	711	889	453	792	194	664	18
43	251	992	996	955	740	888	482	790	223	661	17
44	280	992	.03025	954	769	886	511	788	252	659	16
45	309	991	054	953	798	885	540	786	281	657	15
46	338	991	083	952	827	883	569	784	310	654	14
47	367	991	112	952	856	882	598	782	339	652	13
48	396	990	141	951	885	881	627	780	368	649	12
49	425	990	170	950	914	879	656	778	397	647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	**10**
51	483	989	228	948	972	876	714	774	455	642	9
52	513	989	257	947	.05001	875	743	772	484	639	8
53	542	988	286	946	030	873	773	770	513	637	7
54	571	988	316	945	059	872	802	768	542	635	6
55	600	987	345	944	088	870	831	766	571	632	5
56	629	987	374	943	117	869	860	764	600	630	4
57	658	986	403	942	146	867	889	762	629	627	3
58	687	986	432	941	175	866	918	760	658	625	2
59	716	985	461	940	205	864	947	758	687	622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	**0**
	N. cos.	N. sine	N. cos.	N. sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	
′	89°		88°		87°		86°		85°		′

Natural Tangents and Cotangents

′	0°		1°		2°		3°		4°		′
	Tang	Cotg	Tang	Cotg	Tang	Cotg	Tang	Cotg	Tang	Cotg	
0	.0000	Infinite	.0175	57.2900	.0349	28.6363	.0524	19.0811	.0699	14.3007	**60**
1	03	3437.75	77	56.3506	52	3994	27	18.9755	.0702	2411	**59**
2	06	1718.87	80	55.4415	55	1664	30	8711	05	1821	**58**
3	09	1145.92	83	54.5613	58	27.9372	33	7678	08	1235	**57**
4	12	859.436	86	53.7086	61	7117	36	6656	11	0655	**56**
5	15	687.549	89	52.8821	64	4899	39	5645	14	0079	**55**
6	17	572.957	92	0807	67	2715	42	4645	17	13.9507	**54**
7	20	491.106	95	51.3032	70	0566	44	3655	20	8940	**53**
8	23	429.718	98	50.5485	73	26.8450	47	2677	23	8378	**52**
9	26	381.971	.0201	49.8157	75	6367	50	1708	26	7821	**51**
10	.0029	343.774	.0204	49.1039	.0378	26.4316	.0553	18.0750	.0729	13.7267	**50**
11	32	312.521	07	48.4121	81	2296	56	17.9802	31	6719	49
12	35	286.478	09	47.7395	84	0307	59	8863	34	6174	48
13	38	264.441	12	0853	87	25.8348	62	7934	37	5634	47
14	41	245.552	15	46.4489	90	6418	65	7015	40	5098	46
15	44	229.182	18	45.8294	93	4517	68	6106	43	4566	45
16	47	214.858	21	2261	96	2644	71	5205	46	4039	44
17	49	202.219	24	44.6386	99	0798	74	4314	49	3515	43
18	52	190.984	27	0661	.0402	24.8978	77	3432	52	2996	42
19	55	180.932	30	43.5081	05	7185	80	2558	55	2480	41
20	.0058	171.885	.0233	42.9641	.0407	24.5418	.0582	17.1693	.0758	13.1969	**40**
21	61	163.700	36	4335	10	3675	85	0837	61	1461	39
22	64	156.259	39	41.9158	13	1957	88	16.9990	64	0958	38
23	67	149.465	41	4106	16	0263	91	9150	67	0458	37
24	70	143.237	44	40.9174	19	23.8593	94	8319	69	12.9962	36
25	73	137.507	47	4358	22	6945	97	7496	72	9469	35
26	76	132.219	50	39.9655	25	5321	.0600	6681	75	8981	34
27	79	127.321	53	5059	28	3718	03	5874	78	8496	33
28	81	122.774	56	0568	31	2137	06	5075	81	8014	32
29	84	118.540	59	38.6177	34	0577	09	4283	84	7536	31
30	.0087	114.589	.0262	38.1885	.0437	22.9038	.0612	16.3499	.0787	12.7062	**30**
31	90	110.892	65	37.7686	40	7519	15	2722	90	6591	29
32	93	107.426	68	3579	42	6020	17	1952	93	6124	28
33	96	104.171	71	36.9560	45	4541	20	1190	96	5660	27
34	99	101.107	74	5627	48	3081	23	0435	99	5199	26
35	.0102	98.2179	76	1776	51	1640	26	15.9687	.0802	4742	25
36	05	95.4895	79	35.8006	54	0217	29	8945	05	4288	24
37	08	92.9085	82	4313	57	21.8813	32	8211	08	3838	23
38	11	90.4633	85	0695	60	7426	35	7483	10	3390	22
39	13	88.1436	88	34.7151	63	6056	38	6762	13	2946	21
40	.0116	85.9398	.0291	34.3678	.0466	21.4704	.0641	15.6048	.0816	12.2505	**20**
41	19	83.8435	94	0273	69	3369	44	5340	19	2067	19
42	22	81.8470	97	33.6935	72	2049	47	4638	22	1632	18
43	25	79.9434	.0300	3662	75	0747	50	3943	25	1201	17
44	28	78.1263	03	0452	77	20.9460	53	3254	28	0772	16
45	31	76.3900	06	32.7303	80	8188	55	2571	31	0346	15
46	34	74.7292	08	4213	83	6932	58	1893	34	11.9923	14
47	37	73.1390	11	1181	86	5691	61	1222	37	9504	13
48	40	71.6151	14	31.8205	89	4465	64	0557	40	9087	12
49	43	70.1533	17	5284	92	3253	67	14.9898	43	8673	11
50	.0145	68.7501	.0320	31.2416	.0495	20.2056	.0670	14.9244	.0846	11.8262	**10**
51	48	67.4019	23	30.9599	98	0872	73	8596	49	7853	9
52	51	66.1055	26	6833	.0501	19.9702	76	7954	51	7448	8
53	54	64.8580	29	4116	04	8546	79	7317	54	7045	7
54	57	63.6567	32	1446	07	7403	82	6685	57	6645	6
55	60	62.4992	35	29.8823	09	6273	85	6059	60	6248	5
56	63	61.3829	38	6245	12	5156	88	5438	63	5853	4
57	66	60.3058	40	3711	15	4051	90	4823	66	5461	3
58	69	59.2659	43	1220	18	2959	93	4212	69	5072	2
59	72	58.2612	46	28.8771	21	1879	96	3607	72	4685	1
60	.0175	57.2900	.0349	28.6363	.0524	19.0811	.0699	14.3007	.0875	11.4301	**0**
	Cotg	Tang	Cotg	Tang	Cotg	Tang	Cotg	Tang	Cotg	Tang	
′	89°		88°		87°		86°		85°		′

Natural Sines and Cosines

′	5°		6°		7°		8°		9°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	**60**
1	745	617	482	449	216	251	946	023	672	764	59
2	774	614	511	446	245	248	975	019	701	760	58
3	803	612	540	443	274	244	.14004	015	730	755	57
4	831	609	569	440	302	240	033	011	758	751	56
5	860	607	597	437	331	237	061	006	787	746	55
6	889	604	626	434	360	233	090	002	816	741	54
7	918	602	655	431	389	230	119	.98998	845	737	53
8	947	599	684	428	418	226	148	994	873	732	52
9	976	596	713	424	447	222	177	990	902	728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	**50**
11	034	591	771	418	504	215	234	982	959	718	49
12	063	588	800	415	533	211	263	978	988	714	48
13	092	586	829	412	562	208	292	973	.16017	709	47
14	121	583	858	409	591	204	320	969	046	704	46
15	150	580	887	406	620	200	349	965	074	700	45
16	179	578	916	402	649	197	378	961	103	695	44
17	208	575	945	399	678	193	407	957	132	690	43
18	237	572	973	396	706	189	436	953	160	686	42
19	266	570	.11002	393	735	186	464	948	189	681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	**40**
21	324	564	060	386	793	178	522	940	246	671	39
22	353	562	089	383	822	175	551	936	275	667	38
23	382	559	118	380	851	171	580	931	304	662	37
24	411	556	147	377	880	167	608	927	333	657	36
25	440	553	176	374	908	163	637	923	361	652	35
26	469	551	205	370	937	160	666	919	390	648	34
27	498	548	234	367	966	156	695	914	419	643	33
28	527	545	263	364	995	152	723	910	447	638	32
29	556	542	291	360	.13024	148	752	906	476	633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	**30**
31	614	537	349	354	081	141	810	897	533	624	29
32	642	534	378	351	110	137	838	893	562	619	28
33	671	531	407	347	139	133	867	889	591	614	27
34	700	528	436	344	168	129	896	884	620	609	26
35	729	526	465	341	197	125	925	880	648	604	25
36	758	523	494	337	226	122	954	876	677	600	24
37	787	520	523	334	254	118	982	871	706	595	23
38	816	517	552	331	283	114	.15011	867	734	590	22
39	845	514	580	327	312	110	040	863	763	585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	**20**
41	903	508	638	320	370	102	097	854	820	575	19
42	932	506	667	317	399	098	126	849	849	570	18
43	961	503	696	314	427	094	155	845	878	565	17
44	990	500	725	310	456	091	184	841	906	561	16
45	.10019	497	754	307	485	087	212	836	935	556	15
46	048	494	783	303	514	083	241	832	964	551	14
47	077	491	812	300	543	079	270	827	992	546	13
48	106	488	840	297	572	075	299	823	.17021	541	12
49	135	485	869	293	600	071	327	818	050	536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	**10**
51	192	479	927	286	658	063	385	809	107	526	9
52	221	476	956	283	687	059	414	805	136	521	8
53	250	473	985	279	716	055	442	800	164	516	7
54	279	470	.12014	276	744	051	471	796	193	511	6
55	308	467	043	272	773	047	500	791	222	506	5
56	337	464	071	269	802	043	529	787	250	501	4
57	366	461	100	265	831	039	557	782	279	496	3
58	395	458	129	262	860	035	586	778	308	491	2
59	424	455	158	258	889	031	615	773	336	486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	84°		83°		82°		81°		80°		′

Natural Tangents and Cotangents

′	5°		6°		7°		8°		9°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.0875	11.4301	.1051	9.5144	.1228	8.1443	.1405	7.1154	.1584	6.3138	**60**
1	78	11.3919	54	9.4878	31	248	08	004	87	019	59
2	81	540	57	614	34	054	11	7.0855	90	6.2901	58
3	84	163	60	352	37	8.0860	14	706	93	783	57
4	87	11.2789	63	090	40	667	17	558	96	666	56
5	90	417	66	9.3831	43	476	20	410	99	549	55
6	92	048	69	572	46	285	23	264	.1602	432	54
7	95	11.1681	72	315	49	095	26	117	05	316	53
8	98	316	75	060	51	7.9906	29	6.9972	08	200	52
9	.0901	11.0954	78	9.2806	54	718	32	827	11	085	51
10	.0904	11.0594	.1080	9.2553	.1257	7.9530	.1435	6.9682	.1614	6.1970	**50**
11	07	237	83	302	60	344	38	538	17	856	49
12	10	10.9882	86	052	63	158	41	395	20	742	48
13	13	529	89	9.1803	66	7.8973	44	252	23	628	47
14	16	178	92	555	69	789	47	110	26	515	46
15	19	10.8829	95	309	72	606	50	6.8969	29	402	45
16	22	483	98	065	75	424	53	828	32	290	44
17	25	139	.1101	9.0821	78	243	56	687	35	178	43
18	28	10.7797	04	579	81	062	59	548	38	066	42
19	31	457	07	338	84	7.7882	62	408	41	6.0955	41
20	.0934	10.7119	.1110	9.0098	.1287	7.7704	.1465	6.8269	.1644	6.0844	**40**
21	36	10.6783	13	8.9860	90	525	68	131	47	734	39
22	39	450	16	623	93	348	71	6.7994	50	624	38
23	42	118	19	387	96	171	74	856	53	514	37
24	45	10.5789	22	152	99	7.6996	77	720	55	405	36
25	48	462	25	8.8919	.1302	821	80	584	58	296	35
26	51	136	28	686	05	647	83	448	61	188	34
27	54	10.4813	31	455	08	473	86	313	64	080	33
28	57	491	33	225	11	301	89	179	67	5.9972	32
29	60	172	36	8.7996	14	129	92	045	70	865	31
30	.0963	10.3854	.1139	8.7769	.1317	7.5958	.1495	6.6912	.1673	5.9758	**30**
31	66	538	42	542	19	787	97	779	76	651	29
32	69	224	45	317	22	618	.1500	646	79	545	28
33	72	10.2913	48	093	25	449	03	514	82	439	27
34	75	602	51	8.6870	28	281	06	383	85	333	26
35	78	294	54	648	31	113	09	252	88	228	25
36	81	10.1988	57	427	34	7.4947	12	122	91	124	24
37	83	683	60	208	37	781	15	6.5992	94	019	23
38	86	381	63	8.5989	40	615	18	863	97	5.8915	22
39	89	080	66	772	43	451	21	734	.1700	811	21
40	.0992	10.0780	.1169	8.5555	.1346	7.4287	.1524	6.5606	.1703	5.8708	**20**
41	95	483	72	340	49	124	27	478	06	605	19
42	98	187	75	126	52	7.3962	30	350	09	502	18
43	.1001	9.9893	78	8.4913	55	800	33	223	12	400	17
44	04	601	81	701	58	639	36	097	15	298	16
45	07	310	84	490	61	479	39	6.4971	18	197	15
46	10	021	87	280	64	319	42	846	21	095	14
47	13	9.8734	89	071	67	160	45	721	24	5.7994	13
48	16	448	92	8.3863	70	002	48	596	27	894	12
49	19	164	95	656	73	7.2844	51	472	30	794	11
50	.1022	9.7882	.1198	8.3450	.1376	7.2687	.1554	6.4348	.1733	5.7694	**10**
51	25	601	.1201	245	79	531	57	225	36	594	9
52	28	322	04	041	82	375	60	103	39	495	8
53	30	044	07	8.2838	85	220	63	6.3980	42	396	7
54	33	9.6768	10	636	88	066	66	859	45	297	6
55	36	493	13	434	91	7.1912	69	737	48	199	5
56	39	220	16	234	94	759	72	617	51	101	4
57	42	9.5949	19	035	97	607	75	496	54	004	3
58	45	679	22	8.1837	99	455	78	376	57	5.6906	2
59	48	411	25	640	.1402	304	81	257	60	809	1
60	.1051	9.5144	.1228	8.1443	.1405	7.1154	.1584	6.3138	.1763	5.6713	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	84°		83°		82°		81°		80°		′

Natural Sines and Cosines

′	10°		11°		12°		13°		14°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	**60**
1	393	476	109	157	820	809	523	430	220	023	59
2	422	471	138	152	848	803	552	424	249	015	58
3	451	466	167	146	877	797	580	417	277	008	57
4	479	461	195	140	905	791	608	411	305	001	56
5	508	455	224	135	933	784	637	404	333	.96994	55
6	537	450	252	129	962	778	665	398	362	987	54
7	565	445	281	124	990	772	693	391	390	980	53
8	594	440	309	118	.21019	766	722	384	418	973	52
9	623	435	338	112	047	760	750	378	446	966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	**50**
11	680	425	395	101	104	748	807	365	503	952	49
12	708	420	423	096	132	742	835	358	531	945	48
13	737	414	452	090	161	735	863	351	559	937	47
14	766	409	481	084	189	729	892	345	587	930	46
15	794	404	509	079	218	723	920	338	615	923	45
16	823	399	538	073	246	717	948	331	644	916	44
17	852	394	566	067	275	711	977	325	672	909	43
18	880	389	595	061	303	705	.23005	318	700	902	42
19	909	383	623	056	331	698	033	311	728	894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	**40**
21	966	373	680	044	388	686	090	298	784	880	39
22	995	368	709	039	417	680	118	291	813	873	38
23	.18023	362	737	033	445	673	146	284	841	866	37
24	052	357	766	027	474	667	175	278	869	858	36
25	081	352	794	021	502	661	203	271	897	851	35
26	109	347	823	016	530	655	231	264	925	844	34
27	138	341	851	010	559	648	260	257	954	837	33
28	166	336	880	004	587	642	288	251	982	829	32
29	195	331	908	.97998	616	636	316	244	.25010	822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	**30**
31	252	320	965	987	672	623	373	230	066	807	29
32	281	315	994	981	701	617	401	223	094	800	28
33	309	310	.20022	975	729	611	429	217	122	793	27
34	338	304	051	969	758	604	458	210	151	786	26
35	367	299	079	963	786	598	486	203	179	778	25
36	395	294	108	958	814	592	514	196	207	771	24
37	424	288	136	952	843	585	542	189	235	764	23
38	452	283	165	946	871	579	571	182	263	756	22
39	481	277	193	940	899	573	599	176	291	749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	**20**
41	538	267	250	928	956	560	656	162	348	734	19
42	567	261	279	922	985	553	684	155	376	727	18
43	595	256	307	916	.22013	547	712	148	404	719	17
44	624	250	336	910	041	541	740	141	432	712	16
45	652	245	364	905	070	534	769	134	460	705	15
46	681	240	393	899	098	528	797	127	488	697	14
47	710	234	421	893	126	521	825	120	516	690	13
48	738	229	450	887	155	515	853	113	545	682	12
49	767	223	478	881	183	508	882	106	573	675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	**10**
51	824	212	535	869	240	496	938	093	629	660	9
52	852	207	563	863	268	489	966	086	657	653	8
53	881	201	592	857	297	483	995	079	685	645	7
54	910	196	620	851	325	476	.24023	072	713	638	6
55	938	190	649	845	353	470	051	065	741	630	5
56	967	185	677	839	382	463	079	058	769	623	4
57	995	179	706	833	410	457	108	051	798	615	3
58	.19024	174	734	827	438	450	136	044	826	608	2
59	052	168	763	821	467	444	164	037	854	600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	79°		78°		77°		76°		75°		′

Natural Tangents and Cotangents

′	10°		11°		12°		13°		14°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.1763	5.6713	.1944	5.1446	.2126	4.7046	.2309	4.3315	.2493	4.0108	60
1	66	617	47	366	29	4.6979	12	257	96	058	59
2	69	521	50	286	32	912	15	200	99	009	58
3	72	425	53	207	35	845	18	143	.2503	3.9959	57
4	75	329	56	128	38	779	21	086	06	910	56
5	78	234	59	049	41	712	24	029	09	861	55
6	81	140	62	5.0970	44	646	27	4.2972	12	812	54
7	84	045	65	892	47	580	30	916	15	763	53
8	87	5.5951	68	814	50	514	33	859	18	714	52
9	90	857	71	736	53	448	36	803	21	665	51
10	.1793	5.5764	.1974	5.0658	.2156	4.6382	.2339	4.2747	.2524	3.9617	50
11	96	671	77	581	59	317	42	691	27	568	49
12	99	578	80	504	62	252	45	635	30	520	48
13	.1802	485	83	427	65	187	49	580	33	471	47
14	05	393	86	350	68	122	52	524	37	423	46
15	08	301	89	273	71	057	55	468	40	375	45
16	11	209	92	197	74	4.5993	58	413	43	327	44
17	14	118	95	121	77	928	61	358	46	279	43
18	17	026	98	045	80	864	64	303	49	232	42
19	20	5.4936	.2001	4.9969	83	800	67	248	52	184	41
20	.1823	5.4845	.2004	4.9894	.2186	4.5736	.2370	4.2193	.2555	3.9136	40
21	26	755	07	819	89	673	73	139	58	089	39
22	29	665	10	744	93	609	76	084	61	042	38
23	32	575	13	669	96	546	79	030	64	3.8995	37
24	35	486	16	594	99	483	82	4.1976	68	947	36
25	38	397	19	520	.2202	420	85	922	71	900	35
26	41	308	22	446	05	357	88	868	74	854	34
27	44	219	25	372	08	294	92	814	77	807	33
28	47	131	28	298	11	232	95	760	80	760	32
29	50	043	31	225	14	169	98	706	83	714	31
30	.1853	5.3955	.2035	4.9152	.2217	4.5107	.2401	4.1653	.2586	3.8667	30
31	56	868	38	078	20	045	04	600	89	621	29
32	59	781	41	006	23	4.4983	07	547	92	575	28
33	62	694	44	4.8933	26	922	10	493	95	528	27
34	65	607	47	860	29	860	13	441	99	482	26
35	68	521	50	788	32	799	16	388	.2602	436	25
36	71	435	53	716	35	737	19	335	05	391	24
37	74	349	56	644	38	676	22	282	08	345	23
38	77	263	59	573	41	615	25	230	11	299	22
39	80	178	62	501	44	555	28	178	14	254	21
40	.1883	5.3093	.2065	4.8430	.2247	4.4494	.2432	4.1126	.2617	3.8208	20
41	87	008	68	359	51	434	35	074	20	163	19
42	90	5.2924	71	288	54	374	38	022	23	118	18
43	93	839	74	218	57	313	41	4.0970	27	073	17
44	96	755	77	147	60	253	44	918	30	028	16
45	99	672	80	077	63	194	47	867	33	3.7983	15
46	.1902	588	83	007	66	134	50	815	36	938	14
47	05	505	86	4.7937	69	075	53	764	39	893	13
48	08	422	89	867	72	015	56	713	42	848	12
49	11	339	92	798	75	4.3956	59	662	45	804	11
50	.1914	5.2257	.2095	4.7729	.2278	4.3897	.2462	4.0611	.2648	3.7760	10
51	17	174	98	659	81	838	65	560	51	715	9
52	20	092	.2101	591	84	779	69	509	55	671	8
53	23	011	04	522	87	721	72	459	58	627	7
54	26	5.1929	07	453	90	662	75	408	61	583	6
55	29	848	10	385	93	604	78	358	64	539	5
56	32	767	13	317	96	546	81	308	67	495	4
57	35	686	16	249	99	488	84	257	70	451	3
58	38	606	19	181	.2303	430	87	207	73	408	2
59	41	526	23	114	06	372	90	158	76	364	1
60	.1944	5.1446	.2126	4.7046	.2309	4.3315	.2493	4.0108	.2679	3.7321	0
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	79°		78°		77°		76°		75°		′

Natural Sines and Cosines

′	15°		16°		17°		18°		19°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.25882	.96593	27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	**60**
1	910	585	592	118	265	622	929	097	584	542	59
2	938	578	620	110	293	613	957	088	612	533	58
3	966	570	648	102	321	605	985	079	639	523	57
4	994	562	676	094	348	596	.31012	070	667	514	56
5	.26022	555	704	086	376	588	040	061	694	504	55
6	050	547	731	078	404	579	068	052	722	495	54
7	079	540	759	070	432	571	095	043	749	485	53
8	107	532	787	062	460	562	123	033	777	476	52
9	135	524	815	054	487	554	151	024	804	466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	**50**
11	191	509	871	037	543	536	206	006	859	447	49
12	219	502	899	029	571	528	233	.94997	887	438	48
13	247	494	927	021	599	519	261	988	914	428	47
14	275	486	955	013	626	511	289	979	942	418	46
15	303	479	983	005	654	502	316	970	969	409	45
16	331	471	.28011	.95997	682	493	344	961	997	399	44
17	359	463	039	989	710	485	372	952	.33024	390	43
18	387	456	067	981	737	476	399	943	051	380	42
19	415	448	095	972	765	467	427	933	079	370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	**40**
21	471	433	150	956	821	450	482	915	134	351	39
22	500	425	178	948	849	441	510	906	161	342	38
23	528	417	206	940	876	433	537	897	189	332	37
24	556	410	234	931	904	424	565	888	216	322	36
25	584	402	262	923	932	415	593	878	244	313	35
26	612	394	290	915	960	407	620	869	271	303	34
27	640	386	318	907	987	398	648	860	298	293	33
28	668	379	346	898	.30015	389	675	851	326	284	32
29	696	371	374	890	043	380	703	842	353	274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	**30**
31	752	355	429	874	098	363	758	823	408	254	29
32	780	347	457	865	126	354	786	814	436	245	28
33	808	340	485	857	154	345	813	805	463	235	27
34	836	332	513	849	182	337	841	795	490	225	26
35	864	324	541	841	209	328	868	786	518	215	25
36	892	316	569	832	237	319	896	777	545	206	24
37	920	308	597	824	265	310	923	768	573	196	23
38	948	301	625	816	292	301	951	758	600	186	22
39	976	293	652	807	320	293	979	749	627	176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	**20**
41	032	277	708	791	376	275	034	730	682	157	19
42	060	269	736	782	403	266	061	721	710	147	18
43	088	261	764	774	431	257	089	712	737	137	17
44	116	253	792	766	459	248	116	702	764	127	16
45	144	246	820	757	486	240	144	693	792	118	15
46	172	238	847	749	514	231	171	684	819	108	14
47	200	230	875	740	542	222	199	674	846	098	13
48	228	222	903	732	570	213	227	665	874	088	12
49	256	214	931	724	597	204	254	656	901	078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	**10**
51	312	198	987	707	653	186	309	637	956	058	9
52	340	190	.29015	698	680	177	337	627	983	049	8
53	368	182	042	690	708	168	364	618	.34011	039	7
54	396	174	070	681	736	159	392	609	038	029	6
55	424	166	098	673	763	150	419	599	065	019	5
56	452	158	126	664	791	142	447	590	093	009	4
57	480	150	154	656	819	133	474	580	120	.93999	3
58	508	142	182	647	846	124	502	571	147	989	2
59	536	134	209	639	874	115	529	561	175	979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	74°		73°		72°		71°		70°		′

Natural Tangents and Cotangents

′	15°		16°		17°		18°		19°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.2679	3.7321	.2867	3.4874	.3057	3.2709	.3249	3.0777	.3443	2.9042	**60**
1	83	277	71	836	60	675	52	746	47	015	59
2	86	234	74	798	64	641	56	716	50	2.8987	58
3	89	191	77	760	67	607	59	686	53	960	57
4	92	148	80	722	70	573	62	655	56	933	56
5	95	105	83	684	73	539	65	625	60	905	55
6	98	062	86	646	76	506	69	595	63	878	54
7	.2701	019	90	608	80	472	72	565	66	851	53
8	04	3.6976	93	570	83	438	75	535	69	824	52
9	08	933	96	533	86	405	78	505	73	797	51
10	.2711	3.6891	.2899	3.4495	.3089	3.2371	.3281	3.0475	.3476	2.8770	**50**
11	14	848	.2902	458	92	338	85	445	79	743	49
12	17	806	05	420	96	305	88	415	82	716	48
13	20	764	08	383	99	272	91	385	86	689	47
14	23	722	12	346	.3102	238	94	356	89	662	46
15	26	680	15	308	05	205	98	326	92	636	45
16	29	638	18	271	08	172	.3301	296	95	609	44
17	33	596	21	234	11	139	04	267	99	582	43
18	36	554	24	197	15	106	07	237	.3502	556	42
19	39	512	27	160	18	073	10	208	05	529	41
20	.2742	3.6470	.2931	3.4124	.3121	3.2041	.3314	3.0178	.3508	2.8502	**40**
21	45	429	34	087	24	008	17	149	12	476	39
22	48	387	37	050	27	3.1975	20	120	15	449	38
23	51	346	40	014	31	943	23	090	18	423	37
24	54	305	43	3.3977	34	910	27	061	22	397	36
25	58	264	46	941	37	878	30	032	25	370	35
26	61	222	49	904	40	845	33	003	28	344	34
27	64	181	53	868	43	813	36	2.9974	31	318	33
28	67	140	56	832	47	780	39	945	35	291	32
29	70	100	59	796	50	748	43	916	38	265	31
30	.2773	3.6059	.2962	3.3759	.3153	3.1716	.3346	2.9887	.3541	2.8239	**30**
31	76	018	65	723	56	684	49	858	44	213	29
32	80	3.5978	68	687	59	652	52	829	48	187	28
33	83	937	72	652	63	620	56	800	51	161	27
34	86	897	75	616	66	588	59	772	54	135	26
35	89	856	78	580	69	556	62	743	58	109	25
36	92	816	81	544	72	524	65	714	61	083	24
37	95	776	84	509	75	492	69	686	64	057	23
38	98	736	87	473	79	460	72	657	67	032	22
39	.2801	696	91	438	82	429	75	629	71	006	21
40	.2805	3.5656	.2994	3.3402	.3185	3.1397	.3378	2.9600	.3574	2.7980	**20**
41	08	616	97	367	88	366	82	572	77	955	19
42	11	576	.3000	332	91	334	85	544	81	929	18
43	14	536	03	297	95	303	88	515	84	903	17
44	17	497	06	261	98	271	91	487	87	878	16
45	20	457	10	226	.3201	240	95	459	90	852	15
46	23	418	13	191	04	209	98	431	94	827	14
47	27	379	16	156	07	178	.3401	403	97	801	13
48	30	339	19	122	11	146	04	375	.3600	776	12
49	33	300	22	087	14	115	08	347	04	751	11
50	.2836	3.5261	.3026	3.3052	.3217	3.1084	.3411	2.9319	.3607	2.7725	**10**
51	39	222	29	017	20	053	14	291	10	700	9
52	42	183	32	3.2983	23	022	17	263	13	675	8
53	45	144	35	948	27	3.0991	21	235	17	650	7
54	49	105	38	914	30	961	24	208	20	625	6
55	52	067	41	879	33	930	27	180	23	600	5
56	55	028	45	845	36	899	30	152	27	575	4
57	58	3.4989	48	811	40	868	34	125	30	550	3
58	61	951	51	777	43	838	37	097	33	525	2
59	64	912	54	743	46	807	40	070	36	500	1
60	.2867	3.4874	.3057	3.2709	.3249	3.0777	.3443	2.9042	.3640	2.7475	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	74°		73°		72°		71°		70°		′

Natural Sines and Cosines

′	20°		21°		22°		23°		24°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	**60**
1	229	959	864	348	488	707	100	039	700	343	59
2	257	949	891	337	515	697	127	028	727	331	58
3	284	939	918	327	542	686	153	016	753	319	57
4	311	929	945	316	569	675	180	005	780	307	56
5	339	919	973	306	595	664	207	.91994	806	295	55
6	366	909	.36000	295	622	653	234	982	833	283	54
7	393	899	027	285	649	642	260	971	860	272	53
8	421	889	054	274	676	631	287	959	886	260	52
9	448	879	081	264	703	620	314	948	913	248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	**50**
11	503	859	135	243	757	598	367	925	966	224	49
12	530	849	162	232	784	587	394	914	992	212	48
13	557	839	190	222	811	576	421	902	.41019	200	47
14	584	829	217	211	838	565	448	891	045	188	46
15	612	819	244	201	865	554	474	879	072	176	45
16	639	809	271	190	892	543	501	868	098	164	44
17	666	799	298	180	919	532	528	856	125	152	43
18	694	789	325	169	946	521	555	845	151	140	42
19	721	779	352	159	973	510	581	833	178	128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	**40**
21	775	759	406	137	.38026	488	635	810	231	104	39
22	803	748	434	127	053	477	661	799	257	092	38
23	830	738	461	116	080	466	688	787	284	080	37
24	857	728	488	106	107	455	715	775	310	068	36
25	884	718	515	095	134	444	741	764	337	056	35
26	912	708	542	084	161	432	768	752	363	044	34
27	939	698	569	074	188	421	795	741	390	032	33
28	966	688	596	063	215	410	822	729	416	020	32
29	993	677	623	052	241	399	848	718	443	008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	**30**
31	048	657	677	031	295	377	902	694	496	984	29
32	075	647	704	020	322	366	928	683	522	972	28
33	102	637	731	010	349	355	955	671	549	960	27
34	130	626	758	.92999	376	343	982	660	575	948	26
35	157	616	785	988	403	332	.40008	648	602	936	25
36	184	606	812	978	430	321	035	636	628	924	24
37	211	596	839	967	456	310	062	625	655	911	23
38	239	585	867	956	483	299	088	613	681	899	22
39	266	575	894	945	510	287	115	601	707	887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	**20**
41	320	555	948	924	564	265	168	578	760	863	19
42	347	544	975	913	591	254	195	566	787	851	18
43	375	534	.37002	902	617	243	221	555	813	839	17
44	402	524	029	892	644	231	248	543	840	826	16
45	429	514	056	881	671	220	275	531	866	814	15
46	456	503	083	870	698	209	301	519	892	802	14
47	484	493	110	859	725	198	328	508	919	790	13
48	511	483	137	849	752	186	355	496	945	778	12
49	538	472	164	838	778	175	381	484	972	766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	**10**
51	592	452	218	816	832	152	434	461	.42024	741	9
52	619	441	245	805	859	141	461	449	051	729	8
53	647	431	272	794	886	130	488	437	077	717	7
54	674	420	299	784	912	119	514	425	104	704	6
55	701	410	326	773	939	107	541	414	130	692	5
56	728	400	353	762	966	096	567	402	156	680	4
57	755	389	380	751	993	085	594	390	183	668	3
58	782	379	407	740	.39020	073	621	378	209	655	2
59	810	368	434	729	046	062	647	366	235	643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	69°		68°		67°		66°		65°		′

Natural Tangents and Cotangents

′	20°		21°		22°		23°		24°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.3640	2.7475	.3839	2.6051	.4040	2.4751	.4245	2.3559	.4452	2.2460	**60**
1	43	50	42	28	44	30	48	39	56	43	59
2	46	25	45	06	47	09	52	20	59	25	58
3	50	00	49	2.5983	50	2.4689	55	01	63	08	57
4	53	2.7376	52	61	54	68	58	2.3483	66	2.2390	56
5	56	51	55	38	57	48	62	64	70	73	55
6	59	26	59	16	61	27	65	45	73	55	54
7	63	02	62	2.5893	64	06	69	26	77	38	53
8	66	2.7277	65	71	67	2.4586	72	07	80	20	52
9	69	53	69	48	71	66	76	2.3388	84	03	51
10	.3673	2.7228	.3872	2.5826	.4074	2.4545	.4279	2.3369	.4487	2.2286	**50**
11	76	04	75	04	78	25	83	51	91	68	49
12	79	2.7179	79	2.5782	81	04	86	32	94	51	48
13	83	55	82	59	84	2.4484	89	13	98	34	47
14	86	30	85	37	88	64	93	2.3294	.4501	16	46
15	89	06	89	15	91	43	96	76	05	2.2199	45
16	92	2.7082	92	2.5693	95	23	.4300	57	08	82	44
17	96	58	95	71	98	03	03	38	12	65	43
18	99	34	99	49	.4101	2.4383	07	20	15	48	42
19	.3702	09	.3902	27	05	62	10	01	19	30	41
20	.3706	2.6985	.3906	2.5605	.4108	2.4342	.4314	2.3183	.4522	2.2113	**40**
21	09	61	09	2.5583	11	22	17	64	26	2.2096	39
22	12	37	12	61	15	02	20	46	29	79	38
23	16	13	16	39	18	2.4282	24	27	33	62	37
24	19	2.6889	19	17	22	62	27	09	36	45	36
25	22	65	22	2.5495	25	42	31	2.3090	40	28	35
26	26	41	26	73	29	22	34	72	43	11	34
27	29	18	29	52	32	02	38	53	47	2.1994	33
28	32	2.6794	32	30	35	2.4182	41	35	50	77	32
29	36	70	36	08	39	62	45	17	54	60	31
30	.3739	2.6746	.3939	2.5386	.4142	2.4142	.4348	2.2998	.4557	2.1943	**30**
31	42	23	42	65	46	22	52	80	61	26	29
32	45	2.6699	46	43	49	02	55	62	64	09	28
33	49	75	49	22	52	2.4083	59	44	68	2.1892	27
34	52	52	53	00	56	63	62	25	71	76	26
35	55	28	56	2.5279	59	43	65	07	75	59	25
36	59	05	59	57	63	23	69	2.2889	78	42	24
37	62	2.6581	63	36	66	04	72	71	82	25	23
38	65	58	66	14	69	2.3984	76	53	85	08	22
39	69	34	69	2.5193	73	64	79	35	89	2.1792	21
40	.3772	2.6511	.3973	2.5172	.4176	2.3945	.4383	2.2817	.4592	2.1775	**20**
41	75	2.6488	76	50	80	25	86	2.2799	96	58	19
42	79	64	79	29	83	06	90	81	99	42	18
43	82	41	83	08	87	2.3886	93	63	.4603	25	17
44	85	18	86	2.5086	90	67	97	45	07	08	16
45	89	2.6395	90	65	93	47	.4400	27	10	2.1692	15
46	92	71	93	44	97	28	04	09	14	75	14
47	95	48	96	23	.4200	08	07	2.2691	17	59	13
48	99	25	.4000	02	04	2.3789	11	73	21	42	12
49	.3802	02	03	2.4981	07	70	14	55	24	25	11
50	.3805	2.6279	.4006	2.4960	.4210	2.3750	.4417	2.2637	.4628	2.1609	**10**
51	09	56	10	39	14	31	21	20	31	2.1592	9
52	12	33	13	18	17	12	24	02	35	76	8
53	15	10	17	2.4897	21	2.3693	28	2.2584	38	60	7
54	19	2.6187	20	76	24	73	31	66	42	43	6
55	22	65	23	55	28	54	35	49	45	27	5
56	25	42	27	34	31	35	38	31	49	10	4
57	29	19	30	13	34	16	42	13	52	2.1494	3
58	32	2.6096	33	2.4792	38	2.3597	45	2.2496	56	78	2
59	35	74	37	72	41	78	49	78	60	61	1
60	.3839	2.6051	.4040	2.4751	.4245	2.3559	.4452	2.2460	.4663	2.1445	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	69°		68°		67°		66°		65°		′

Natural Sines and Cosines

′	25°		26°		27°		28°		29°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	**60**
1	288	618	863	867	425	087	973	281	506	448	59
2	315	606	889	854	451	074	999	267	532	434	58
3	341	594	916	841	477	061	.47024	254	557	420	57
4	367	582	942	828	503	048	050	240	583	406	56
5	394	569	968	816	529	035	076	226	608	391	55
6	420	557	994	803	554	021	101	213	634	377	54
7	446	545	.44020	790	580	008	127	199	659	363	53
8	473	532	046	777	606	.88995	153	185	684	349	52
9	499	520	072	764	632	981	178	172	710	335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	**50**
11	552	495	124	739	684	955	229	144	761	306	49
12	578	483	151	726	710	942	255	130	786	292	48
13	604	470	177	713	736	928	281	117	811	278	47
14	631	458	203	700	762	915	306	103	837	264	46
15	657	446	229	687	787	902	332	089	862	250	45
16	683	433	255	674	813	888	358	075	888	235	44
17	709	421	281	662	839	875	383	062	913	221	43
18	736	408	307	649	865	862	409	048	938	207	42
19	762	396	333	636	891	848	434	034	964	193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	**40**
21	815	371	385	610	942	822	486	006	.49014	164	39
22	841	358	411	597	968	808	511	.87993	040	150	38
23	867	346	437	584	994	795	537	979	065	136	37
24	894	334	464	571	.46020	782	562	965	090	121	36
25	920	321	490	558	046	768	588	951	116	107	35
26	946	309	516	545	072	755	614	937	141	093	34
27	972	296	542	532	097	741	639	923	166	079	33
28	999	284	568	519	123	728	665	909	192	064	32
29	.43025	271	594	506	149	715	690	896	217	050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	**30**
31	077	246	646	480	201	688	741	868	268	021	29
32	104	233	672	467	226	674	767	854	293	007	28
33	130	221	698	454	252	661	793	840	318	.86993	27
34	156	208	724	441	278	647	818	826	344	978	26
35	182	196	750	428	304	634	844	812	369	964	25
36	209	183	776	415	330	620	869	798	394	949	24
37	235	171	802	402	355	607	895	784	419	935	23
38	261	158	828	389	381	593	920	770	445	921	22
39	287	146	854	376	407	580	946	756	470	906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	**20**
41	340	120	906	350	458	553	997	729	521	878	19
42	366	108	932	337	484	539	.48022	715	546	863	18
43	392	095	958	324	510	526	048	701	571	849	17
44	418	082	984	311	536	512	073	687	596	834	16
45	445	070	.45010	298	561	499	099	673	622	820	15
46	471	057	036	285	587	485	124	659	647	805	14
47	497	045	062	272	613	472	150	645	672	791	13
48	523	032	088	259	639	458	175	631	697	777	12
49	549	019	114	245	664	445	201	617	723	762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	**10**
51	602	.89994	166	219	716	417	252	589	773	733	9
52	628	981	192	206	742	404	277	575	798	719	8
53	654	968	218	193	767	390	303	561	824	704	7
54	680	956	243	180	793	377	328	546	849	690	6
55	706	943	269	167	819	363	354	532	874	675	5
56	733	930	295	153	844	349	379	518	899	661	4
57	759	918	321	140	870	336	405	504	924	646	3
58	785	905	347	127	896	322	430	490	950	632	2
59	811	892	373	114	921	308	456	476	975	617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	64°		63°		62°		61°		60°		′

Natural Tangents and Cotangents

′	25°		26°		27°		28°		29°		′
	Tang.	**Cotg.**	**Tang.**	**Cotg.**	**Tang.**	**Cotg.**	**Tang.**	**Cotg.**	**Tang.**	**Cotg.**	
0	.4663	2.1445	.4877	2.0503	.5095	1.9626	.5317	1.8807	.5543	1.8040	**60**
1	67	29	81	2.0488	99	12	21	1.8794	47	28	59
2	70	13	85	73	.5103	1.9598	25	81	51	16	58
3	74	2.1396	88	58	06	84	28	68	55	03	57
4	77	80	92	43	10	70	32	55	58	1.7991	56
5	81	64	95	28	14	56	36	41	62	79	55
6	84	48	99	13	17	42	40	28	66	66	54
7	88	32	.4903	2.0398	21	28	43	15	70	54	53
8	91	15	06	83	25	14	47	02	74	42	52
9	95	2.1299	10	68	28	00	51	1.8689	77	30	51
10	.4699	2.1283	.4913	2.0353	.5132	1.9486	.5354	1.8676	.5581	1.7917	**50**
11	.4702	67	17	38	36	72	58	63	85	05	49
12	06	51	21	23	39	58	62	50	89	1.7893	48
13	09	35	24	08	43	44	66	37	93	81	47
14	13	19	28	2.0293	47	30	69	24	96	68	46
15	16	03	31	78	50	16	73	11	.5600	56	45
16	20	2.1187	35	63	54	02	77	1.8598	04	44	44
17	23	71	39	48	58	1.9388	81	85	08	32	43
18	27	55	42	33	61	75	84	72	12	20	42
19	31	39	46	19	65	61	88	59	16	08	41
20	.4734	2.1123	.4950	2.0204	.5169	1.9347	.5392	1.8546	.5619	1.7796	**40**
21	38	07	53	2.0189	72	33	96	33	23	83	39
22	41	2.1092	57	74	76	19	99	20	27	71	38
23	45	76	60	60	80	06	.5403	07	31	59	37
24	48	60	64	45	84	1.9292	07	1.8495	35	47	36
25	52	44	68	30	87	78	11	82	39	35	35
26	55	28	71	15	91	65	15	69	42	23	34
27	59	13	75	01	95	51	18	56	46	11	33
28	63	2.0997	79	2.0086	98	37	22	43	50	1.7699	32
29	66	81	82	72	.5202	23	26	30	54	87	31
30	.4770	2.0965	.4986	2.0057	.5206	1.9210	.5430	1.8418	.5658	1.7675	**30**
31	73	50	89	42	09	1.9196	33	05	62	63	29
32	77	34	93	28	13	83	37	1.8392	65	51	28
33	80	18	97	13	17	69	41	79	69	39	27
34	84	03	.5000	1.9999	20	55	45	67	73	27	26
35	88	2.0887	04	84	24	42	48	54	77	15	25
36	91	72	08	70	28	28	52	41	81	03	24
37	95	56	11	55	32	15	56	29	85	1.7591	23
38	98	40	15	41	35	01	60	16	88	79	22
39	.4802	25	19	26	39	1.9088	64	03	92	67	21
40	.4806	2.0809	.5022	1.9912	.5243	1.9074	.5467	1.8291	.5696	1.7556	**20**
41	09	2.0794	26	1.9897	46	61	71	78	.5700	44	19
42	13	78	29	83	50	47	75	65	04	32	18
43	16	63	33	68	54	34	79	53	08	20	17
44	20	48	37	54	58	20	82	40	12	08	16
45	23	32	40	40	61	07	86	28	15	1.7496	15
46	27	17	44	25	65	1.8993	90	15	19	85	14
47	31	01	48	11	69	80	94	02	23	73	13
48	34	2.0686	51	1.9797	72	67	98	1.8190	27	61	12
49	38	71	55	82	76	53	.5501	77	31	49	11
50	.4841	2.0655	.5059	1.9768	.5280	1.8940	.5505	1.8165	.5735	1.7437	**10**
51	45	40	62	54	84	27	09	52	39	26	9
52	49	25	66	40	87	13	13	40	43	14	8
53	52	09	70	25	91	00	17	27	46	02	7
54	56	2.0594	73	11	95	1.8887	20	15	50	1.7391	6
55	59	79	77	1.9697	98	73	24	03	54	79	5
56	63	64	81	83	.5302	60	28	1.8090	58	67	4
57	67	49	84	69	06	47	32	78	62	55	3
58	70	33	88	54	10	34	35	65	66	44	2
59	74	18	92	40	13	20	39	53	70	32	1
60	.4877	2.0503	.5095	1.9626	.5317	1.8807	.5543	1.8040	.5774	1.7321	**0**
	Cotg.	**Tang.**	**Cotg.**	**Tang.**	**Cotg.**	**Tang.**	**Cotg.**	**Tang.**	**Cotg.**	**Tang.**	
′	64°		63°		62°		61°		60°		′

Natural Sines and Cosines

′	30°		31°		32°		33°		34°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	**60**
1	025	588	529	702	.53017	789	488	851	943	887	59
2	050	573	554	687	041	774	513	835	968	871	58
3	076	559	579	672	066	759	537	819	992	855	57
4	101	544	604	657	091	743	561	804	.56016	839	56
5	126	530	628	642	115	728	586	788	040	822	55
6	151	515	653	627	140	712	610	772	064	806	54
7	176	501	678	612	164	697	635	756	088	790	53
8	201	486	703	597	189	681	659	740	112	773	52
9	227	471	728	582	214	666	683	724	136	757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	**50**
11	277	442	778	551	263	635	732	692	184	724	49
12	302	427	803	536	288	619	756	676	208	708	48
13	327	413	828	521	312	604	781	660	232	692	47
14	352	398	852	506	337	588	805	645	256	675	46
15	377	384	877	491	361	573	829	629	280	659	45
16	403	369	902	476	386	557	854	613	305	643	44
17	428	354	927	461	411	542	878	597	329	626	43
18	453	340	952	446	435	526	902	581	353	610	42
19	478	325	977	431	460	511	927	565	377	593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	**40**
21	528	295	026	401	509	480	975	533	425	561	39
22	553	281	051	385	534	464	999	517	449	544	38
23	578	266	076	370	558	448	.55024	501	473	528	37
24	603	251	101	355	583	433	048	485	497	511	36
25	628	237	126	340	607	417	072	469	521	495	35
26	654	222	151	325	632	402	097	453	545	478	34
27	679	207	175	310	656	386	121	437	569	462	33
28	704	192	200	294	681	370	145	421	593	446	32
29	729	178	225	279	705	355	169	405	617	429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	**30**
31	779	148	275	249	754	324	218	373	665	396	29
32	804	133	299	234	779	308	242	356	689	380	28
33	829	119	324	218	804	292	266	340	713	363	27
34	854	104	349	203	828	277	291	324	736	347	26
35	879	089	374	188	853	261	315	308	760	330	25
36	904	074	399	173	877	245	339	292	784	314	24
37	929	059	423	157	902	230	363	276	808	297	23
38	954	045	448	142	926	214	388	260	832	281	22
39	979	030	473	127	951	198	412	244	856	264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	**20**
41	029	000	522	096	.54000	167	460	212	904	231	19
42	054	.85985	547	081	024	151	484	195	928	214	18
43	079	970	572	066	049	135	509	179	952	198	17
44	104	956	597	051	073	120	533	163	976	181	16
45	129	941	621	035	097	104	557	147	.57000	165	15
46	154	926	646	020	122	088	581	131	024	148	14
47	179	911	671	005	146	072	605	115	047	132	13
48	204	896	696	.84989	171	057	630	098	071	115	12
49	229	881	720	974	195	041	654	082	095	098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	**10**
51	279	851	770	943	244	009	702	050	143	065	9
52	304	836	794	928	269	.83994	726	034	167	048	8
53	329	821	819	913	293	978	750	017	191	032	7
54	354	806	844	897	317	962	775	001	215	015	6
55	379	792	869	882	342	946	799	.82985	238	.81999	5
56	404	777	893	866	366	930	823	969	262	982	4
57	429	762	918	851	391	915	847	953	286	965	3
58	454	747	943	836	415	899	871	936	310	949	2
59	479	732	967	820	440	883	895	920	334	932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	59°		58°		57°		56°		55°		′

Natural Tangents and Cotangents

′	30°		31°		32°		33°		34°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.5774	1.7321	.6009	1.6643	.6249	1.6003	.6494	1.5399	.6745	1.4826	**60**
1	77	09	13	32	53	1.5993	98	89	49	16	59
2	81	1.7297	17	21	57	83	.6502	79	54	07	58
3	85	86	20	10	61	72	06	69	58	1.4798	57
4	89	74	24	1.6599	65	62	11	59	62	88	56
5	93	62	28	88	69	52	15	50	66	79	55
6	97	51	32	77	73	41	19	40	71	70	54
7	.5801	39	36	66	77	31	23	30	75	61	53
8	05	28	40	55	81	21	27	20	79	51	52
9	08	16	44	45	85	11	31	11	83	42	51
10	.5812	1.7205	.6048	1.6534	.6289	1.5900	.6536	1.5301	.6787	1.4733	**50**
11	16	1.7193	52	23	93	1.5890	40	1.5291	92	24	49
12	20	82	56	12	97	80	44	82	96	15	48
13	24	70	60	01	.6301	69	48	72	.6800	05	47
14	28	59	64	1.6490	05	59	52	62	05	1.4696	46
15	32	47	68	79	10	49	56	53	09	87	45
16	36	36	72	69	14	39	60	43	13	78	44
17	40	24	76	58	18	29	65	33	17	69	43
18	44	13	80	47	22	18	69	24	22	1.4659	42
19	47	02	84	36	26	08	73	14	26	50	41
20	.5851	1.7090	.6088	1.6426	.6330	1.5798	.6577	1.5204	.6830	1.4641	**40**
21	55	79	92	15	34	88	81	1.5195	34	32	39
22	59	67	96	04	38	78	85	85	39	23	38
23	63	56	.6100	1.6393	42	68	90	75	43	14	37
24	67	45	04	83	46	57	94	66	47	05	36
25	71	33	08	72	50	47	98	56	51	1.4596	35
26	75	22	12	61	54	37	.6602	47	56	86	34
27	79	11	16	51	58	27	06	37	60	77	33
28	83	1.6999	20	40	63	17	10	27	64	68	32
29	87	88	24	29	67	07	15	18	69	59	31
30	.5890	1.6977	.6128	1.6319	.6371	1.5697	.6619	1.5108	.6873	1.4550	**30**
31	94	65	32	08	75	87	23	1.5099	77	41	29
32	98	54	36	1.6297	79	77	27	89	81	32	28
33	.5902	43	40	87	83	67	31	80	86	23	27
34	06	32	44	76	87	57	36	70	90	14	26
35	10	20	48	65	91	47	40	61	94	05	25
36	14	09	52	55	95	37	44	51	99	1.4496	24
37	18	1.6898	56	44	99	27	48	42	.6903	87	23
38	22	87	60	34	.6403	17	52	32	07	78	22
39	26	75	64	23	08	07	57	23	11	69	21
40	.5930	1.6864	.6168	1.6212	.6412	1.5597	.6661	1.5013	.6916	1.4460	**20**
41	34	53	72	02	16	87	65	04	20	51	19
42	38	42	76	1.6191	20	77	69	1.4994	24	42	18
43	42	31	80	81	24	67	73	85	29	33	17
44	45	20	84	70	28	57	78	75	33	24	16
45	49	08	88	60	32	47	82	66	37	15	15
46	53	1.6797	92	49	36	37	86	57	42	06	14
47	57	86	96	39	40	27	90	47	46	1.4397	13
48	61	75	.6200	28	45	17	94	38	50	88	12
49	65	64	04	18	49	07	99	28	54	79	11
50	.5969	1.6753	.6208	1.6107	.6453	1.5497	.6703	1.4919	.6959	1.4370	**10**
51	73	42	12	1.6097	57	87	07	10	63	61	9
52	77	31	16	87	61	77	11	00	67	52	8
53	81	20	20	76	65	68	15	1.4891	72	44	7
54	85	09	24	66	69	58	20	82	76	35	6
55	89	1.6698	28	55	73	48	24	72	80	26	5
56	93	87	33	45	78	38	28	63	85	17	4
57	97	76	37	34	82	28	32	54	89	08	3
58	.6001	65	41	24	86	18	37	44	93	1.4299	2
59	05	54	45	14	90	08	41	35	98	90	1
60	.6009	1.6643	.6249	1.6003	.6494	1.5399	.6745	1.4826	.7002	1.4281	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	59°		58°		57°		56°		55°		′

′	35°		36°		37°		38°		39°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	**60**
1	381	899	802	885	205	846	589	783	955	696	59
2	405	882	826	867	228	829	612	765	977	678	58
3	429	865	849	850	251	811	635	747	.63000	660	57
4	453	848	873	833	274	793	658	729	022	641	56
5	477	832	896	816	298	776	681	711	045	623	55
6	501	815	920	799	321	758	704	694	068	605	54
7	524	798	943	782	344	741	726	676	090	586	53
8	548	782	967	765	367	723	749	658	113	568	52
9	572	765	990	748	390	706	772	640	135	550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	**50**
11	619	731	037	713	437	671	818	604	180	513	49
12	643	714	061	696	460	653	841	586	203	494	48
13	667	698	084	679	483	635	864	568	225	476	47
14	691	681	108	662	506	618	887	550	248	458	46
15	715	664	131	644	529	600	909	532	271	439	45
16	738	647	154	627	553	583	932	514	293	421	44
17	762	631	178	610	576	565	955	496	316	402	43
18	786	614	201	593	599	547	978	478	338	384	42
19	810	597	225	576	622	530	.62001	460	361	366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	**40**
21	857	563	272	541	668	494	046	424	406	329	39
22	881	546	295	524	691	477	069	405	428	310	38
23	904	530	318	507	714	459	092	387	451	292	37
24	928	513	342	489	738	441	115	369	473	273	36
25	952	496	365	472	761	424	138	351	496	255	35
26	976	479	389	455	784	406	160	333	518	236	34
27	999	462	412	438	807	388	183	315	540	218	33
28	58023	445	436	420	830	371	206	297	563	199	32
29	047	428	459	403	853	353	229	279	585	181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	**30**
31	094	395	506	368	899	318	274	243	630	144	29
32	118	378	529	351	922	300	297	225	653	125	28
33	141	361	552	334	945	282	320	206	675	107	27
34	165	344	576	316	968	264	342	188	698	088	26
35	189	327	599	299	991	247	365	170	720	070	25
36	212	310	622	282	.61015	229	388	152	742	051	24
37	236	293	646	264	038	211	411	134	765	033	23
38	260	276	669	247	061	193	433	116	787	014	22
39	283	259	693	230	084	176	456	098	810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	**20**
41	330	225	739	195	130	140	502	061	854	959	19
42	354	208	763	178	153	122	524	043	877	940	18
43	378	191	786	160	176	105	547	025	899	921	17
44	401	174	809	143	199	087	570	007	922	903	16
45	425	157	832	125	222	069	592	.77988	944	884	15
46	449	140	856	108	245	051	615	970	966	866	14
47	472	123	879	091	268	033	638	952	989	847	13
48	496	106	902	073	291	016	660	934	.64011	828	12
49	519	089	926	056	314	.78998	683	916	033	810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	**10**
51	567	055	972	021	360	962	728	879	078	772	9
52	590	038	995	003	383	944	751	861	100	754	8
53	614	021	.60019	.79986	406	926	774	843	123	735	7
54	637	004	042	968	429	908	796	824	145	717	6
55	661	.80987	065	951	451	891	819	806	167	698	5
56	684	970	089	934	474	873	842	788	190	679	4
57	708	953	112	916	497	855	864	769	212	661	3
58	731	936	135	899	520	837	887	751	234	642	2
59	755	919	158	881	543	819	909	733	256	623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	54°		53°		52°		51°		50°		′

Natural Tangents and Cotangents

′	35°		36°		37°		38°		39°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.7002	1.4281	.7265	1.3764	.7536	1.3270	.7813	1.2799	.8098	1.2349	**60**
1	06	73	70	55	40	62	18	92	.8103	42	59
2	11	64	74	47	45	54	22	84	07	34	58
3	15	55	79	39	49	46	27	76	12	27	57
4	19	46	83	30	54	38	32	69	17	20	56
5	24	37	88	22	58	30	36	61	22	12	55
6	28	29	92	13	63	22	41	53	27	05	54
7	32	20	97	05	68	14	46	46	32	1.2298	53
8	37	11	.7301	1.3697	72	06	50	38	36	90	52
9	41	02	06	88	77	1.3198	55	31	41	83	51
10	.7046	1.4193	.7310	1.3680	.7581	1.3190	.7860	1.2723	.8146	1.2276	**50**
11	50	85	14	72	86	82	65	15	51	68	49
12	54	76	19	63	90	75	69	08	56	61	48
13	59	67	23	55	95	67	74	00	61	54	47
14	63	58	28	47	.7600	59	79	1.2693	65	47	46
15	67	50	32	38	04	51	83	85	70	39	45
16	72	41	37	30	09	43	88	77	75	32	44
17	76	32	41	22	13	35	93	70	80	25	43
18	80	24	46	13	18	27	98	62	85	18	42
19	85	15	50	05	23	19	.7902	55	90	10	41
20	.7089	1.4106	.7355	1.3597	.7627	1.3111	.7907	1.2647	.8195	1.2203	**40**
21	94	1.4097	59	88	32	03	12	40	99	1.2196	39
22	98	89	64	80	36	1.3095	16	32	.8204	89	38
23	.7102	80	68	72	41	87	21	24	09	81	37
24	07	71	73	64	46	79	26	17	14	74	36
25	11	63	77	55	50	72	31	09	19	67	35
26	15	54	82	47	55	64	35	02	24	60	34
27	20	45	86	39	59	56	40	1.2594	29	53	33
28	24	37	91	31	64	48	45	87	34	45	32
29	29	28	95	22	69	40	50	79	38	38	31
30	.7133	1.4019	.7400	1.3514	.7673	1.3032	.7954	1.2572	.8243	1.2131	**30**
31	37	11	04	06	78	24	59	64	48	24	29
32	42	02	09	1.3498	83	17	64	57	53	17	28
33	46	1.3994	13	90	87	09	69	49	58	09	27
34	51	85	18	81	92	01	73	42	63	02	26
35	55	76	22	73	96	1.2993	78	34	68	1.2095	25
36	59	68	27	65	.7701	85	83	27	73	88	24
37	64	59	31	57	06	77	88	19	78	81	23
38	68	51	36	49	10	70	92	12	83	74	22
39	73	42	40	40	15	62	97	04	87	66	21
40	.7177	1.3934	.7445	1.3432	.7720	1.2954	.8002	1.2497	.8292	1.2059	**20**
41	81	25	49	24	24	46	07	89	97	52	19
42	86	16	54	16	29	38	12	82	.8302	45	18
43	90	08	58	08	34	31	16	75	07	38	17
44	95	1.3899	63	00	38	23	21	67	12	31	16
45	99	91	67	1.3392	43	15	26	60	17	24	15
46	.7203	82	72	84	47	07	31	52	22	17	14
47	08	74	76	75	52	00	35	45	27	09	13
48	12	65	81	67	57	1.2892	40	37	32	02	12
49	17	57	85	59	61	84	45	30	37	1.1995	11
50	.7221	1.3848	.7490	1.3351	.7766	1.2876	.8050	1.2423	.8342	1.1988	**10**
51	26	40	95	43	71	69	55	15	46	81	9
52	30	31	99	35	75	61	59	08	51	74	8
53	34	23	.7504	27	80	53	64	01	56	67	7
54	39	14	08	19	85	46	69	1.2393	61	60	6
55	43	06	13	11	89	38	74	86	66	53	5
56	48	1.3798	17	03	94	30	79	78	71	46	4
57	52	89	22	1.3295	99	22	83	71	76	39	3
58	57	81	26	87	.7803	15	88	64	81	32	2
59	61	72	31	78	08	07	93	56	86	25	1
60	.7265	1.3764	.7536	1.3270	.7813	1.2799	.8098	1.2349	.8391	1.1918	**0**
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	54°		53°		52°		51°		50°		′

Natural Sines and Cosines

′	40°		41°		42°		43°		44°		′
	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	N.sine	N. cos.	
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	**60**
1	301	586	628	452	935	295	221	116	487	914	59
2	323	567	650	433	956	276	242	096	508	894	58
3	346	548	672	414	978	256	264	076	529	873	57
4	368	530	694	395	999	237	285	056	549	853	56
5	390	511	716	375	.67021	217	306	036	570	833	55
6	412	492	738	356	043	198	327	016	591	813	54
7	435	473	759	337	064	178	349	.72996	612	792	53
8	457	455	781	318	086	159	370	976	633	772	52
9	479	436	803	299	107	139	391	957	654	752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	**50**
11	524	398	847	261	151	100	434	917	696	711	49
12	546	380	869	241	172	080	455	897	717	691	48
13	568	361	891	222	194	061	476	877	737	671	47
14	590	342	913	203	215	041	497	857	758	650	46
15	612	323	935	184	237	022	518	837	779	630	45
16	635	304	956	165	258	002	539	817	800	610	44
17	657	286	978	146	280	.73983	561	797	821	590	43
18	679	267	.66000	126	301	963	582	777	842	569	42
19	701	248	022	107	323	944	603	757	862	549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	**40**
21	746	210	066	069	366	904	645	717	904	508	39
22	768	192	088	050	387	885	666	697	925	488	38
23	790	173	109	030	409	865	688	677	946	468	37
24	812	154	131	011	430	846	709	657	966	447	36
25	834	135	153	.74992	452	826	730	637	987	427	35
26	856	116	175	973	473	806	751	617	.70008	407	34
27	878	097	197	953	495	787	772	597	029	386	33
28	901	078	218	934	516	767	793	577	049	366	32
29	923	059	240	915	538	747	814	557	070	345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	**30**
31	967	022	284	876	580	708	857	517	112	305	29
32	989	003	306	857	602	688	878	497	132	284	28
33	.65011	.75984	327	838	623	669	899	477	153	264	27
34	033	965	349	818	645	649	920	457	174	243	26
35	055	946	371	799	666	629	941	437	195	223	25
36	077	927	393	780	688	610	962	417	215	203	24
37	100	908	414	760	709	590	983	397	236	182	23
38	122	889	436	741	730	570	.69004	377	257	162	22
39	144	870	458	722	752	551	025	357	277	141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	**20**
41	188	832	501	683	795	511	067	317	319	100	19
42	210	813	523	664	816	491	088	297	339	080	18
43	232	794	545	644	837	472	109	277	360	059	17
44	254	775	566	625	859	452	130	257	381	039	16
45	276	756	588	606	880	432	151	236	401	019	15
46	298	738	610	586	901	413	172	216	422	.70998	14
47	320	719	632	567	923	393	193	196	443	978	13
48	342	700	653	548	944	373	214	176	463	957	12
49	364	680	675	528	965	353	235	156	484	937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	**10**
51	408	642	718	489	.68008	314	277	116	525	896	9
52	430	623	740	470	029	294	298	095	546	875	8
53	452	604	762	451	051	274	319	075	567	855	7
54	474	585	783	431	072	254	340	055	587	834	6
55	496	566	805	412	093	234	361	035	608	813	5
56	518	547	827	392	115	215	382	015	628	793	4
57	540	528	848	373	136	195	403	.71995	649	772	3
58	562	509	870	353	157	175	424	974	670	752	2
59	584	490	891	334	179	155	445	955	690	731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	**0**
	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	N.cos.	N. sine	
′	49°		48°		47°		46°		45°		′

Natural Tangents and Cotangents

′	40°		41°		42°		43°		44°		′
	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	
0	.8391	1.1918	.8693	1.1504	.9004	1.1106	.9325	1.0724	.9657	1.0355	60
1	96	10	98	1.1497	09	00	31	17	63	49	59
2	.8401	03	.8703	90	15	1.1093	36	11	68	43	58
3	06	1.1896	08	83	20	87	41	05	74	37	57
4	11	89	13	77	25	80	47	1.0699	79	31	56
5	16	82	18	70	30	74	52	92	85	25	55
6	21	75	24	63	36	67	58	86	91	19	54
7	26	68	29	56	41	61	63	80	96	13	53
8	31	61	34	50	46	54	69	74	.9702	07	52
9	36	54	39	43	52	48	74	68	08	01	51
10	.8441	1.1847	.8744	1.1436	.9057	1.1041	.9380	1.0661	.9713	1.0295	50
11	46	40	49	30	62	35	85	55	19	89	49
12	51	33	54	23	67	28	91	49	25	83	48
13	56	26	59	16	73	22	96	43	30	77	47
14	61	19	65	10	78	16	.9402	37	36	71	46
15	66	12	70	03	83	09	07	30	42	65	45
16	71	06	75	1.1396	89	03	13	24	47	59	44
17	76	1.1799	80	89	94	1.0996	18	18	53	53	43
18	81	92	85	83	99	90	24	12	59	47	42
19	86	85	90	76	.9105	83	29	06	64	41	41
20	.8491	1.1778	.8796	1.1369	.9110	1.0977	.9435	1.0599	.9770	1.0235	40
21	96	71	.8801	63	15	71	40	93	76	30	39
22	.8501	64	06	56	21	64	46	87	81	24	38
23	06	57	11	49	26	58	51	81	87	18	37
24	11	50	16	43	31	51	57	75	93	12	36
25	16	43	21	36	37	45	62	69	98	06	35
26	21	36	27	29	42	39	68	62	.9804	00	34
27	26	29	32	23	47	32	73	56	10	1.0194	33
28	31	22	37	16	53	26	79	50	16	88	32
29	36	15	42	10	58	19	84	44	21	82	31
30	.8541	1.1708	.8847	1.1303	.9163	1.0913	.9490	1.0538	.9827	1.0176	30
31	46	02	52	1.1296	69	07	95	32	33	70	29
32	51	1.1695	58	90	74	00	.9501	26	38	64	28
33	56	88	63	83	79	1.0894	06	19	44	58	27
34	61	81	68	76	85	88	12	13	50	52	26
35	66	74	73	70	90	81	17	07	56	47	25
36	71	67	78	63	95	75	23	01	61	41	24
37	76	60	84	57	.9201	69	28	1.0495	67	35	23
38	81	53	89	50	06	62	34	89	73	29	22
39	86	47	94	43	12	56	40	83	79	23	21
40	.8591	1.1640	.8899	1.1237	.9217	1.0850	.9545	1.0477	.9884	1.0117	20
41	96	33	.8904	30	22	43	51	70	90	11	19
42	.8601	26	10	24	28	37	56	64	96	05	18
43	06	19	15	17	33	31	62	58	.9902	1.0099	17
44	11	12	20	11	39	24	67	52	07	94	16
45	17	06	25	04	44	18	73	46	13	88	15
46	22	1.1599	31	1.1197	49	12	78	40	19	82	14
47	27	92	36	91	55	05	84	34	25	76	13
48	32	85	41	84	60	1.0799	90	28	30	70	12
49	37	78	46	78	66	93	95	22	36	64	11
50	.8642	1.1571	.8952	1.1171	.9271	1.0786	.9601	1.0416	.9942	1.0058	10
51	47	65	57	65	76	80	06	10	48	52	9
52	52	58	62	58	82	74	12	04	54	47	8
53	57	51	67	52	87	68	18	1.0398	59	41	7
54	62	44	72	45	93	61	23	92	65	35	6
55	67	38	78	39	98	55	29	85	71	29	5
56	72	31	83	32	.9303	49	34	79	77	23	4
57	78	24	88	26	09	42	40	73	83	17	3
58	83	17	94	19	14	36	46	67	88	12	2
59	88	10	99	13	20	30	51	61	94	06	1
60	.8693	1.1504	.9004	1.1106	.9325	1.0724	.9657	1.0355	1.0000	1.0000	0
	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	Cotg.	Tang.	
′	49°		48°		47°		46°		45°		′

APPENDIX D

SQUARES OF NUMBERS

Table I. Squares of Numbers

	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.020	1.040	1.061	1.082	1.102	1.124	1.145	1.166	1.188
.1	1.210	1.232	1.254	1.277	1.300	1.322	1.346	1.369	1.392	1.416
.2	1.440	1.464	1.488	1.513	1.538	1.562	1.588	1.613	1.638	1.664
.3	1.690	1.716	1.742	1.769	1.796	1.822	1.850	1.877	1.904	1.932
.4	1.960	1.988	2.016	2.045	2.074	2.102	2.132	2.161	2.190	2.220
.5	2.250	2.280	2.310	2.341	2.372	2.402	2.434	2.465	2.496	2.528
.6	2.560	2.592	2.624	2.657	2.690	2.722	2.756	2.789	2.822	2.856
.7	2.890	2.924	2.958	2.993	3.028	3.062	3.098	3.133	3.168	3.204
.8	3.240	3.276	3.312	3.349	3.386	3.422	3.460	3.497	3.534	3.572
.9	3.610	3.648	3.686	3.725	3.764	3.802	3.842	3.881	3.920	3.960
2.0	4.000	4.040	4.080	4.121	4.162	4.202	4.244	4.285	4.326	4.368
.1	4.410	4.452	4.494	4.537	4.580	4.622	4.666	4.709	4.752	4.796
.2	4.840	4.884	4.928	4.973	5.018	5.062	5.108	5.153	5.198	5.244
.3	5.290	5.336	5.382	5.429	5.476	5.522	5.570	5.617	5.664	5.712
.4	5.760	5.808	5.856	5.905	5.954	6.002	6.052	6.101	6.150	6.200
.5	6.250	6.300	6.350	6.401	6.452	6.502	6.554	6.605	6.656	6.708
.6	6.760	6.812	6.864	6.917	6.970	7.022	7.076	7.129	7.182	7.236
.7	7.290	7.344	7.398	7.453	7.508	7.562	7.618	7.673	7.728	7.784
.8	7.840	7.896	7.952	8.009	8.066	8.122	8.180	8.237	8.294	8.352
.9	8.410	8.468	8.526	8.585	8.644	8.702	8.762	8.821	8.880	8.940
3.0	9.000	9.060	9.120	9.181	9.242	9.302	9.364	9.425	9.486	9.548
.1	9.610	9.672	9.734	9.797	9.860	9.922	9.986	10.05	10.11	10.18
.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82
.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36
.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13
.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73
.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27
.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16
.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07
.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91
.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14

Table I. Squares of Numbers

	0	1	2	3	4	5	6	7	8	9
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69
.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12
.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
.7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10
.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10
.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14
.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26
.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30
.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17
.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97
.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
.8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

ANSWERS

CHAPTER I

1-1. (*a*) 25 cps
(*b*) 0.01 sec
(*c*) 0.000111 sec
1/9000 sec

1-2. (*a*) 1/2*f* sec
(*b*) 1/360*f* sec

1-3. 4 time-degrees

1-4. 270 time-degrees

1-5. 50 cps

1-6. 500 rpm

1-7. 48 poles

1-8. 40 cps

1-9. 1050 rpm minimum
2100 rpm maximum

1-10. (*a*) 99 volts
(*b*) − 99 volts

1-11. 50.8 amp

1-12. (*a*) 50 volts
(*b*) 50 volts
(*c*) 0 volt
(*d*) − 50 volts
(*e*) − 50 volts

1-13. (*a*) 377 radians per sec
(*b*) 0.786 radian
(*c*) 106.1

1-14. (*a*) 50 cps
(*b*) 0.0025 sec

1-19. 151.5 volts

1-20. 28.2 amp

1-22. 28.28 amp

1-23. 162.6 volts

1-24. 10 ohms impedance
5 ohms resistance

1-25. 157 ohms impedance
40.8 ohms resistance

1-27. 56.5 volts

1-28. (*a*) 0.0795 henry
(*b*) 4.51×10^7 flux linkages
(*c*) 10^5 maxwells

1-29. (*a*) 0.127 henry
(*b*) 1.08×10^8 flux linkages

1-30. 4.8 amp

1-31. 6.85 amp

1-32. 1.1568 amp

1-33. 0.01358 coulomb at 25 cps
0.01358 coulomb at 60 cps

1-34. 95.5 volts

1-35. 18.8 mfd

1-36. 127.6 ohms
20.8 mfd

1-37. (*a*) 66.2 ohms
(*b*) 6.64 amp
(*c*) 659 volts

1-41. (*a*) 0.728 lag
(*b*) 43° 18′

1-42. 982 va

1-43. 100 volts

1-44. 0.00219 sec

1-45. 0.00061 sec

1-46. 141 amp

1-47. 0.025 sec

1-48. 10^6 cps

1-49. 0.458 space-degree

1-50. 45 space-degrees

1-52. 6.96 ohms

1-53. 165.2 volts

1-54. (*a*) 234 ohms
(*b*) 0.939 amp
(*c*) 0 watt

1-55. (*a*) 0.0012 henry
(*b*) 0.452 ohm

1-56. (*a*) 57.2 ohms
(*b*) 46.3 mfd
(*c*) 0 watt
(*d*) 0.8 amp

1-57. 37.6 ohms at 30 cps
50.2 ohms at 40 cps
62.7 ohms at 50 cps

1-58. 176.3 ohms at 30 cps
132.2 ohms at 40 cps
105.8 ohms at 50 cps

1-59. (*a*) 208 amp
(*b*) 50 kw
(*c*) 45 kw
(*d*) 30 kw

1-60. 0.199 lag

1-61. (*a*) 1150 watts
(*b*) 690 watts
(*c*) 345 watts
(*d*) 118 per cent

CHAPTER II

2-2. 100 volts
36° 52′ lead

2-4. 20.25 amp
30° 44′ lag

2-5. (*a*) 86.6 volts
(*b*) 50 volts

2-6. (*a*) 100 volts
173.2 volts

2-7. (*a*) 84.5 volts
(*b*) 89.3 per cent lag

2-8. 3.26 amp

2-9. (*a*) $22\underline{/224^\circ}$
$22\overline{\backslash 136^\circ}$
(*b*) $128\overline{\backslash 31^\circ\,58'}$
(*c*) $14\underline{/75^\circ\,42'}$
(*d*) $3.24\overline{\backslash 116^\circ 27'}$

2-10. (*a*) $0.064\underline{/134^\circ}$
(*b*) $2.84\underline{/218^\circ\,16'}$
$2.84\overline{\backslash 141^\circ\,44'}$
(*c*) $12.4\underline{/45^\circ\,28'}$
(*d*) $68.4\underline{/328^\circ\,26'}$
$68.4\overline{\backslash 31^\circ\,34'}$

2-11. (*a*) $24 - j\,11$
(*b*) $-\,18 + j\,32$
(*c*) $86.6 + j\,50$

2-12. (*a*) $-\,33 + j\,54$
(*b*) $-\,66 + j\,18$
(*c*) $16 - j\,9$

2-13. (*a*) $14.9 - j\,10.13$
(*b*) $4.24 + j\,1.905$
(*c*) $-\,1150 - j\,1995$

2-14. (*a*) $64.8\underline{/195^\circ\,4'}$
$64.8\overline{\backslash 164^\circ\,56'}$
(*b*) $0.00285\overline{\backslash 33^\circ\,42'}$
(*c*) $191.2\underline{/154^\circ}$

2-15. $46.2\underline{/209^\circ\,52'}$
$46.2\overline{\backslash 150^\circ\,8'}$

2-16. $192\underline{/138^\circ\,39'}$

2-17. $19.68\underline{/246^\circ\,48'}$
$19.68\overline{\backslash 113^\circ\,12'}$

2-18. $274.3\overline{\backslash 17^\circ\,4'}$

2-19. $16.3\overline{\backslash 27^\circ\,14'}$ amp

2-20. $82.2\overline{\backslash 44^\circ\,21'}$ volts

2-21. $15.4\underline{/18^\circ\,30'}$ amp

2-22. (*a*) $35\underline{/180^\circ}$
(*b*) $403\overline{\backslash 106^\circ}$
(*c*) $880\underline{/204^\circ}$
$880\overline{\backslash 156^\circ}$

2-23. (*a*) $82 + j\,39$
(*b*) $-\,33 + j\,58$
(*c*) $-\,344 + j\,92$

2-24. (*a*) $27.5\overline{\backslash 105^\circ}$
(*b*) $22\underline{/46^\circ}$

2-25. (*a*) $1.4 + j\,2.8$
(*b*) $3.25 + j\,4.6$

2-26. $89.5\overline{\backslash 22^\circ}$ ohms
83 ohms
33.5 ohms capacitive

2-27. (*a*) $0.0806\overline{\backslash 48^\circ}$
(*b*) $0.147\underline{/34^\circ\,20'}$
(*c*) $0.0513\underline{/132^\circ\,35'}$

2-28. (*a*) $0.118 - j\,0.0294$
(*b*) $-\,0.0216 + j\,0.0297$
(*c*) $-\,0.07 - j\,0.1082$

2-29. $0.1658\underline{/31^\circ\,9'}$ amp

2-30. (*a*) $36\underline{/50^\circ}$
(*b*) $32\overline{\backslash 150^\circ}$
(*c*) $4.5\underline{/27^\circ}$

2-31. (*a*) $64\underline{/90^\circ}$
(*b*) $81\overline{\backslash 60^\circ}$
(*c*) $5.2\overline{\backslash 75^\circ}$

2-32. (*a*) $1264\underline{/163^\circ}$
(*b*) $-12.2 - j\,1.95$
(*c*) $-195 + j\,171$

2-33. (*a*) $32\underline{/20^\circ}$
(*b*) $53.8\underline{/216^\circ\,42'}$
$53.8\overline{\backslash 143^\circ\,18'}$
(*c*) $5.46\overline{\backslash 64^\circ\,12'}$

CHAPTER III

3-1. (*a*) $52.8\underline{/40^\circ\,45'}$ ohms
(*b*) $2.272\overline{\backslash 40^\circ\,45'}$ amp
with *E* on axis of reals

3-2. (*a*) $36.2\underline{/44^\circ}$ ohms
(*b*) $6.08\overline{\backslash 44^\circ}$ amp
with *E* on axis of reals

3-3. $215.8\underline{/90^\circ}$ volts
with *I* on axis of reals

3-4. (*a*) 11.5 ohms
(*b*) $30\underline{/23^\circ\,34'}$ ohms

3-5. $66.4\underline{/49^\circ\,26'}$ ohms

3-6. $4.15\underline{/70^\circ\,5'}$ amp
with *E* on axis of reals

3-7. (*a*) $72\overline{\backslash 90^\circ}$ volts
with *I* on axis of reals
(*b*) 47.8 mfd
(*c*) $222\overline{\backslash 36^\circ\,52'}$ ohms

3-8. $82.6\overline{\backslash 40^\circ\,58'}$ ohms

3-9. 54.2 mfd

3-10. (*a*) $5.52\overline{\backslash 51^\circ\,14'}$ amp
with *E* on axis of reals
(*b*) $138\underline{/0^\circ}$, $415\underline{/90^\circ}$,
$244\overline{\backslash 90^\circ}$ volts
with *I* on axis of reals

3-11. $78\overline{\backslash 35^\circ\,52'}$ volts
with *I* on axis of reals

3-12. $2.83\underline{/10^\circ\,48'}$ amp
with *E* on axis of reals

3-13. (*a*) 53.8 cps

3-14. (*a*) 53.8 cps

3-15. $16\underline{/3^\circ\,30'}$ amp
with E on axis of reals

3-16. (*a*) $27.5\overline{\backslash 3^\circ\,30'}$ ohms
(*b*) 27.44 ohms
(*c*) 1.68 ohms capacitive

3-17. (*a*) $0.0258\overline{\backslash 8^\circ\,42'}$ mho
(*b*) $38.8\underline{/8^\circ\,42'}$ ohms
(*c*) 38.4 ohms
(*d*) 5.9 ohms inductive

3-18. (*a*) $0.153\underline{/26^\circ\,36'}$ mho
(*b*) $6.54\overline{\backslash 26^\circ\,36'}$ ohms
(*c*) 5.85 ohms
(*d*) 2.93 ohms capacitive

3-19. (*a*) 53.8 cps

3-23. 2320 watts

3-24. (*a*) 418 watts
(*b*) 0.867 lag

3-25. 508 watts

3-26. (*a*) 475 watts
(*b*) 0.702 lag

3-27. (*a*) 137.2 watts
(*b*) 0.377 lag

3-28. (*a*) 692 watts
769 watts
(*b*) 1461 watts
(*c*) 0.938 lag

3-29. (*a*) 0.723 lag
43° 42′ lag
(*b*) $52.3\underline{/43^\circ\,42'}$ ohms
(*c*) 38.1 ohms
(*d*) 18.1 ohms
(*e*) 36.4 ohms inductive
(*f*) 0.116 henry
(*g*) 0.45 lag
63° 15′

3-30. (*a*) $24\underline{/78^\circ}$ ohms
(*b*) 23.5 ohms
(*c*) 500 watts

3-31. (*a*) 12,200 volts
(*b*) 0.631 lag
(*c*) 86.1 per cent

3-32. 5.45 ohms

3-33. (*a*) 150 kva
(*b*) 120 kw
(*c*) 75 kw

3-34. 124 mfd

3-35. (*a*) 19.7 kva
(*b*) 0.864 lag
(*c*) 17 kw

3-36. (*a*) 183 kva
(*b*) 175 kw
(*c*) 0.957 lag

3-37. $70\overline{\backslash 56^\circ\ 58'}$ ohms
3-38. $53.2\underline{/29^\circ\ 35'}$ ohms

3-39. 0.707 lag

3-40. (*a*) 0.0357 mfd
(*b*) $543\underline{/0^\circ}$ ohms

3-41. 408 mfd

3-42. (*a*) $169\overline{\backslash 56^\circ\ 30'}$ ohms
(*b*) 93.2 ohms
(*c*) 22.5 mfd

3-43. 1.26 ohms resistance
2.18 ohms reactance

3-44. 11 ohms

3-45. 0.2 millihenry

3-46. 2.69×10^6 cps

3-47. (*a*) 4.18
with *E* on axis of reals
(*b*) 24.2 ohms resistance
10.38 ohms reactance

3-48. (*a*) 0.334 lag
(*b*) 12.2 ohms resistance
34.6 ohms reactance

3-49. 159,000 cps

3-50. 3.91 ohms
97 mfd

3-51. (*a*) 7053 volts
(*b*) 0.79 lag
(*c*) 94.8 per cent

3-52. $74.5\overline{\backslash 34^\circ\ 40'}$ ohms

3-53. (*a*) $85.2\underline{/0^\circ}$ ohms
(*b*) $0.0587\underline{/0^\circ}$ amp
(*c*) 0.1015 watt
(*d*) 0.0372 watt

CHAPTER IV

4-1. 2395 volts
4-3. 83.8 amp
4-4. (*a*) 4160 volts
416 amp
(*b*) 2400 volts
722 amp
4-5. (*a*) 208 volts
625 amp
(*b*) 120 volts
1082 amp
4-6. $44.6\overline{\backslash 51^\circ\ 32'}$ amp
4-7. $16.35\overline{\backslash 135^\circ\ 4'}$ amp
4-8. (*a*) $I_{a'a} = 55.8 - j\,15.5$ amp
$= 58.\overline{\backslash 15^\circ\ 32'}$ amp
$I_{b'b} = -\ 65.1 - j\,63.9$ amp
$= 91.2\overline{\backslash 135^\circ\ 36'}$ amp
$I_{c'c} = 9.3 + j\,79.4$ amp
$= 79.9\underline{/83^\circ\ 20'}$ amp
(*b*) $V_{an} = 4050 + j\,315.5$ volts
$= 4060\underline{/4^\circ\ 28'}$ volts
$V_{bn} = -\ 4770 - j\,2690$ volts
$= 5472\overline{\backslash 150^\circ\ 36'}$ volts
$V_{cn} = -\ 3510 + j\,5350$ volts
$= 6392\underline{/123^\circ\ 20'}$ volts
(*c*) $V_{ab} = 8820 + j\,3006$ volts
$= 9320\underline{/18^\circ\ 53'}$ volts
$V_{bc} = -\ 1260 - j\,8040$ volts
$= 8148\overline{\backslash 98^\circ\ 51'}$ volts
$V_{ca} = -\ 7560 + j\,5034$ volts
$= 9083\underline{/146^\circ\ 22'}$ volts
(*d*) $I_{a'a}Z_G = 265 + j\,450$ volts
$= 522\underline{/59^\circ\ 28'}$ volts

$I_{b'b}Z_G = 403 - j\,715$ volts
$= 821\overline{\backslash 60^\circ\ 36'}$ volts
$I_{c'c}Z_G = -\ 669 + j\,266$ volts
$= 720\underline{/158^\circ\ 20'}$ volts
(e) $I_{a'a}Z_L = 842.4 + j\,217.3$ volts
$= 870\underline{/14^\circ\ 28'}$ volts
$I_{b'b}Z_L = -\ 368 - j\,1318$ volts
$= 1368\overline{\backslash 105^\circ\ 36'}$ volts
$I_{c'c}Z_L = -474.5 + j\,1100$ volts
$= 1198\underline{/113^\circ\ 20'}$ volts

4-9. (a) $I_{nn'} = -\ 14.1 - j\,99$ amp
$= 100\overline{\backslash 98^\circ\ 6'}$ amp
(b) $I_{a'a} = 110.5 - j\,91$ amp
$= 143.1\overline{\backslash 39^\circ\ 28'}$ amp
$I_{b'b} = -\ 68.2 - j\,145.5$ amp
$= 160.7\overline{\backslash 115^\circ\ 6'}$ amp
$I_{c'c} = -\ 56.6 + j\,138$ amp
$= 149.2\underline{/112^\circ\ 18'}$ amp
(c) $V_{an} = 100.8 + j\,9.75$ volts
$= 101.2\underline{/5^\circ\ 32'}$ volts
$V_{bn} = -\ 69.2 - j\,74.2$ volts
$= 101.5\overline{\backslash 133^\circ\ 32'}$ volts
$V_{cn} = -\ 45.3 + j\,110.2$ volts
$= 119.4\underline{/112^\circ\ 18'}$ volts
(d) $I_{a'a}Z_G = 7.73\underline{/28^\circ\ 44'}$ volts
$I_{b'b}Z_G = 8.67\overline{\backslash 46^\circ\ 54'}$ volts
$I_{c'c}Z_G = 806\underline{/180^\circ\ 30'}$ volts
(e) $I_{nn'}Z_n = 12.65\overline{\backslash 79^\circ\ 40'}$ volts
$I_{a'a}Z_L = 10.3\overline{\backslash 5^\circ\ 43'}$ volts
$I_{b'b}Z_L = 11.6\overline{\backslash 81^\circ\ 21'}$ volts
$I_{c'c}Z_L = 10.78\underline{/146^\circ\ 3'}$ volts

4-10. $6.67 + j\,5$ ohms

4-11. $Z_1 = 3.59 + j\,1.07$ ohms
$Z_2 = 2.54 + j\,2.4$ ohms
$Z_3 = 4.17 + j\,0.47$ ohms

4-12. $4.5 - j\,36$ ohms

4-13. $Z_a = 143.4 + j\,75.1$ ohms
$Z_b = 133.3 - j\,21.6$ ohms
$Z_c = 129 - j\,77.6$

4-14. $I_{a'a} = 178.74 - j\,128.16$ amp
$= 220\overline{\backslash 35^\circ\ 41'}$ amp
$I_{b'b} = -\ 12.74 + j\,177.36$ amp
$= 178/94^\circ\ 6'$ amp
$I_{c'c} = -\ 166 - j\,49.2$ amp
$= 173.1\overline{\backslash 163^\circ\ 30'}$ amp

4-15. $I_{a'a} = -\ 85.4 + j\,73$ amp
$= 85.7\underline{/175^\circ\ 6'}$ amp
$I_{b'b} = 44.38 - j\,92.7$ amp
$= 102.8\overline{\backslash 64^\circ\ 25'}$ amp
$I_{c'c} = 41.02 + j\,85.4$ amp
$= 96.7\underline{/63^\circ\ 49'}$ amp

4-16. $I_{a'a} = -\ 92.63 + j\,90.05$ amp
$= 129.2\underline{/135^\circ 48'}$ amp
$I_{b'b} = 85.11 + j\,27.78$ amp
$= 89.53\underline{/18^\circ\ 6'}$ amp
$I_{c'c} = 7.52 - j\,117.83$ amp
$= 118.1\overline{\backslash 86^\circ\ 21'}$ amp

4-17. 0.75 lag

4-18. 36.7 amp

4-19. (a) 11,430 volts
379 amp
6,000 kw
(b) 6,600 volts
656 amp
6,000 kw

4-21. Three-wattmeter method
$P_1 =$ 42,240 watts
$P_2 =$ 61,600 watts
$P_3 =$ 39,600 watts
$P_T =$ 143,440 watts
Two-wattmeter method
$P_1 =$ 88,200 watts
$P_2 =$ 55,200 watts
$P_T =$ 143,400 watts

4-22. Three-wattmeter method
$P_1 =$ 99,600 watts
$P_2 =$ 104,300 watts
$P_3 =$ 101,900 watts
$P_T =$ 305,800 watts
Two-wattmeter method
$P_1 =$ 187,880 watts
$P_2 =$ 117,720 watts
$P_T =$ 305,600 watts

4-23. 0.3733

4-24. (a) 587 amp
(b) 311 volts

4-25. (a) $I_{a'a} = 8.36 + j\,16.6$ amp
$= 18.6/63^\circ\ 16'$ amp

$I_{b'b} = 13.3 - j\,14.2$ amp
$= 19.46\overline{\backslash 46^\circ\,54'}$ amp
$I_{nn'} = 21.66 + j\,2.4$ amp
$= 21.8\underline{/6^\circ\,19'}$ amp

(*b*) $V_{an} = 1.2 + j\,415.6$ volts
$= 415.6\underline{/89^\circ\,50'}$ volts
$V_{bn} = 412.5 - j\,13.5$ volts
$= 412.7\overline{\backslash 1^\circ\,53'}$ volts

(*c*) $I_{a'a}Z_L = -4.92 + j\,23.3$ volts
$= 23.8\underline{/101^\circ\,55'}$ volts
$I_{b'b}Z_L = 24.66 - j\,3.56$ volts
$= 24.9\overline{\backslash 8^\circ\,12'}$ volts

(*d*) $I_{nn'}Z_n = 11.82 + j\,12.27$ volts
$= 17.03\underline{/46^\circ\,1'}$ volts

(*e*) $I_{a'a}Z_G = -8.29 + 8.3$ volts
$= 11.7\underline{/135^\circ}$ volts
$I_{b'b}Z_G = 11.18 + j\,5.14$ volts
$= 12.3\underline{/24^\circ\,45'}$ volts

4-26. $I_{an} = 55 + j\,165$ amp
$= 173.9\underline{/71^\circ\,34'}$ amp
$I_{bn} = 186.5 - j\,103.7$ amp
$= 213.2\overline{\backslash 29^\circ\,3'}$ amp
$I_{nn'} = 241.5 + j\,61.3$ amp
$= 249.2\underline{/14^\circ\,15'}$ amp

4-27. (*a*) 1252 kva
(*b*) 0.81 lag

4-28. (*a*) 128.2 kva
(*b*) 111.5 kw
(*c*) 0.87 lag

4-29. 242 amp

4-30. (*a*) 0.86 lag
(*b*) $P_1 = 2590$ watts
$P_2 = 5274$ watts

4-31. (*a*) 252 amp
11,440 volts
(*b*) 437 amp
6600 volts

4-32. (*a*) 2310 volts
(*b*) $I_{a'a} = 93.8\overline{\backslash 14^\circ\,5'}$ amp
$I_{b'b} = 43.3\overline{\backslash 151^\circ\,47'}$ amp
$I_{c'c} = 43.3\underline{/88^\circ\,13'}$ amp
$I_{nn'} = 54.2\underline{/0^\circ}$ amp.
(Obtained with lagging p.f. and E_{an} on axis of reals)

4-33. $I_{a'a} = 276\overline{\backslash 60^\circ}$ amp
$I_{b'b} = 478\underline{/150^\circ}$ amp
$I_{c'c} = 276\underline{/0^\circ}$ amp
(Obtained with E_{cb} on axis of reals)

4-34. (*a*) $I_{an} = 100.64 - j\,75.48$ amp
$= 125.8\overline{\backslash 36^\circ\,52'}$ amp
$I_{bn} = -8.25 - j\,113.00$ amp
$= 113.2\overline{\backslash 94^\circ\,10'}$ amp
$I_{cn} = -11.45 + j\,156.9$ amp
$= 157.2\underline{/94^\circ\,10'}$ amp
$I_{nn'} = 80.94 - j\,31.58$ amp
$= 86.8\overline{\backslash 21^\circ\,19'}$ amp
(Obtained with E_{an} on axis of reals)
(*b*) 109.4 kw.

4-35. (*a*) $I_{cb} = 309\overline{\backslash 30^\circ\,42'}$ amp
$I_{ac} = 309\overline{\backslash 150^\circ\,42'}$ amp
$I_{ba} = 309\underline{/89^\circ\,18'}$ amp
(*b*) $I_{a'a} = 535\overline{\backslash 120^\circ\,42'}$ amp
$I_{b'b} = 535\underline{/119^\circ\,18'}$ amp
$I_{c'c} = 535\overline{\backslash 0^\circ\,42'}$ amp
(Obtained with E_{cb} on axis of reals)
(*c*) 1825 kw

4-36. $P_1 = 28.8$ kw
$P_2 = 48.2$ kw

4-37. 0.975

4-38. (With E_{an} on axis of reals)
$I_{a'a} = 74.1 - j\,27.7$ amp
$= 79.3\overline{\backslash 20^\circ\,30'}$ amp
$I_{b'b} = 1.9 - j\,52.6$ amp
$= 52.7\overline{\backslash 87^\circ\,54'}$ amp
$I_{nn'} = 76.0 - 80.3$ amp
$= 110\overline{\backslash 46^\circ\,34'}$ amp

4-39. 1456 watts

4-40. 44.73 kva

4-41. (*a*) 1710 kva
(*b*) 0.843 lag
(*c*) 1440 kw

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